

Spinorial geometry, horizons and superconformal symmetry in six dimensions

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Abstract

The spinorial geometry method of solving Killing spinor equations is reviewed as it applies to 6-dimensional (1,0) supergravity. In particular, it is explained how the method is used to identify both the fractions of supersymmetry preserved by and the geometry of all supersymmetric backgrounds. Then two applications are described to systems that exhibit superconformal symmetry. The first is the proof that some 6-dimensional black hole horizons are locally isometric to $AdS_3 \times \Sigma^3$, where Σ^3 is diffeomorphic to S^3 . The second one is a description of all supersymmetric solutions of 6-dimensional (1,0) superconformal theories and in particular of their brane solitons.

1 Introduction

The main purpose of this review article is to describe the spinorial geometry method [1] as it applies into the classification of supersymmetric backgrounds of 6-dimensional (1,0) supergravity theories and then present two applications. One application is an investigation into the geometry of black hole horizons and the other the construction of brane solitons in (1,0)-superconformal theories.

From the very beginning of supersymmetry theories, solutions that preserve some of the supersymmetry of an underlying theory have had a central role in the description of their classical and quantum properties. In supersymmetric gauge theories, such solutions include the solitons and instantons, see eg [2] for a review, which have applications in the understanding of these theories at strong coupling [3, 4]. In the context of supergravity, supersymmetric solutions either serve as backgrounds for compactifications or describe certain classes of black hole solutions, see [5] and [6] for reviews. These results from supergravity theories were later adapted to string theory and M-theory. In addition string theory and M-theory open the arena for new classes of supersymmetric solutions, like those of branes and their intersections, see eg [7, 8] and references within. Such new solutions have been instrumental in the foundation of string and M-theory dualities and as well as in AdS/CFT, see eg [9, 10] for reviews.

Initially, the construction of supersymmetric solutions either in supergravity or in string/M-theory has been centred around an ansatz on the fields motivated by symmetries of the physical object or process under investigation. Such an approach has been very successful and has produced a vast number of solutions many of which have some key applications. However, such an approach is rather limiting as it focuses on particular points of what may be a large set, or moduli space, of similar solutions and it lacks an overview of the possibilities that may be available. Therefore to gain an insight into the structure of string theory and M-theory as well as for many applications in AdS/CFT and black holes a more systematic approach to the construction of supersymmetric solutions is needed.

The problem of classifying the supersymmetric solutions of supergravity theories has been known for sometime. Using twistor methods, P Tod classified the supersymmetric solutions of simple 4-dimensional supergravities [11]. Gauntlett et al in [12] solved KSEs of minimal 5-dimensional supergravity using a technique based on spinor bi-linears, and later this was applied in [13] to solve the Killing spinor equations (KSEs) of D=11 supergravity for one spinor. J Figueroa-O'Farrill and one of the authors classified the maximal supersymmetric solutions of 10- and 11-dimensional supergravities using the integrability conditions of the KSEs [14].

The spinorial geometry method proposed Gillard, Gran and one of the authors in [1] utilizes spinorial techniques and it was originally applied to solve the KSEs of 11-dimensional supergravity. One of its characteristics is that it provides a systematic way to solve the KSEs of supersymmetric theories [15]. The main results of this method have been the solution of the KSEs of IIB and IIA supergravities for one Killing spinor [16, 17], and the solution of KSEs of heterotic [18, 19] and 6-dimensional (1,0) supergravities [20] in all cases, as well as many other applications in other lower dimensional supergravity theories, see eg [21, 22, 23, 24, 25]. In addition, spinorial geometry has been used to classify

all near maximal supersymmetric solutions of IIB [26] and 11-dimensional supergravities [27].

The spinorial geometry method to solving KSEs is based on three ingredients. To describe these ingredients, first observe that all supergravity theories have a local gauge group which includes $Spin(D)$ as a subgroup, where D is the dimension of spacetime. The first ingredient of spinorial geometry is to use the gauge group of a supergravity theory to locally choose representatives of the Killing spinors. These are labeled by the orbits of the gauge group on the space of spinors. The second ingredient is a realization of spinors in terms of forms which simplifies the way that the KSEs act on the spinor representatives, and the third the use of an oscillator basis in the space of spinors which allows the rewriting of the KSEs in terms of a linear system. This linear system has as unknowns components of the fluxes as well as components of the spin connection of the supergravity theory. The linear system is then solved to express some of the fluxes in terms of the geometry and also find the restrictions on the geometry required for the existence of Killing spinors. The latter restrictions are typically expressed as a linear relation between the components of the spin connection. The expressions of the fluxes in terms of the geometry and the conditions on the geometry can be organized in irreducible representations of the isotropy group of the Killing spinors in the gauge group of the supergravity theory.

For the application of spinorial geometry at hand, we shall describe how the spinorial geometry has been used in [20] to solve the KSEs of 6-dimensional (1,0) supergravity coupled to any number of vector, tensor and scalar multiplets [28, 29, 31] in all cases. In particular, we shall describe how all the fractions of supersymmetry preserved by the backgrounds have been identified as well as what is the geometry of the underlying spacetime in all cases.

Furthermore, we shall present two applications of the above results in the context of superconformal systems. One application is the classification of all near horizon geometries of 6-dimensional (1,0) supergravity coupled to tensor and scalar multiplets described in [32]. In particular, we shall show that a class of horizons is isometric to $AdS_3 \times \Sigma^3$, where the universal cover of Σ^3 is diffeomorphic to S^3 , and depending on the geometry of Σ^3 can preserve 2, 4 or 8 supersymmetries.

Another application that we shall demonstrate is the solution of the KSEs of (1,0)-superconformal theories in 6 dimensions [33, 34]. Such theories have been proposed, [35, 36], in the context of finding a Lagrangian description for a multiple M5-brane theory which is conjectured to be the field theory dual of M-theory on $AdS_7 \times S^4$. We shall demonstrate that large classes of such (1,0)-superconformal symmetries have soliton solutions which are expected from the M-brane intersection rules [37, 38].

2 Spinorial geometry

2.1 A paradigm

Before we proceed to apply the spinorial geometry method to solve the KSEs of 6-dimensional supergravity, we shall illustrate how this works in an example. For this

consider the KSE

$$F_{\mu\nu}\Gamma^{\mu\nu}\epsilon = 0 , \quad (2.1)$$

which arises in 6-dimensional Euclidean gauge theory, where ϵ is a spinor, F is a gauge field strength on \mathbb{R}^6 and the gauge indices are suppressed. Solution of this equation means to find the geometric conditions on F such that there is an $\epsilon \neq 0$, called Killing spinor, which solves the above equation.

As we have mentioned the spinorial geometry method proceeds in three steps. First is to identify the orbits of the gauge group of the system on the space of spinors, second is to realize the spinors in terms of forms, and third is to use a basis in the space of spinors to turn the KSEs into a linear system. This system then can be solved to find the conditions of F such that (2.1) has a solution. In practice all steps are related as if one has a convenient realization of spinors as in step 2, then it is more convenient to find the orbits of the gauge group on the space of spinors required in step 1, and to introduce a basis so that step 3 can be carried out. So let us begin with step 2.

2.2 Spinors in terms of forms

Let us consider the spinor representations of $Spin(6)$. These can be constructed by identifying the Dirac representation with the space of forms on \mathbb{C}^3 , $\Lambda^*(\mathbb{C}^3)$. Then a realization of Dirac gamma matrices is

$$\Gamma_i = e_i \wedge + e_i \lrcorner , \quad \Gamma_{3+i} = i(e_i \wedge - e_i \lrcorner) , \quad i = 1, 2, 3 , \quad (2.2)$$

where (e_i) is a Hermitian basis in \mathbb{C}^3 and \lrcorner is the inner derivation operation on $\Lambda^*(\mathbb{C}^3)$ which is adjoint to the wedging. One can verify that the above gamma matrices $(\Gamma_a) = (\Gamma_i, \Gamma_{3+i})$ satisfy the Clifford algebra relations $\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\delta_{ab}$.

The decomposition of forms in even and odd according to their degree, $\Lambda^*(\mathbb{C}^3) = \Lambda^{\text{ev}}(\mathbb{C}^3) \oplus \Lambda^{\text{od}}(\mathbb{C}^3)$, corresponds to the decomposition of the Dirac representation into chiral (Weyl) and anti-chiral (anti-Weyl) representations. The Dirac and chiral representations are complex. There is a real (Majorana) representation of $Spin(6)$ as well identified as the eigenspace of the operator, $R = \Gamma_{456}*$, in $\Lambda^*(\mathbb{C}^3)$ with eigenvalue 1. Observe that $R^2 = 1$ and that R is anti-linear. Real (Majorana) spinors have both chiral and anti-chiral components.

2.3 Orbits of the gauge group and linear system

To identify the gauge group of KSE (2.1), observe that under a $Spin(6)$ transformation of ϵ , the KSE transforms covariantly provided that there is a compensating $SO(6)$ rotation of F . Therefore the gauge group of the KSE (2.1) is $Spin(6)$. Since the solutions ϵ of the KSE are identified up to a gauge transformation, the independent solutions are labeled by the orbits of the gauge group in the space of spinors or the orbits of the gauge group in appropriate number of tensor copies for more than one Killing spinor. For the solution of the KSE, any representative of ϵ in an orbit can be chosen.

To find the orbits of $Spin(6)$ in the space of spinors, it is convenient to use the isomorphism $Spin(6) = SU(4)$. Under this isomorphism, the chiral and anti-chiral representations of $Spin(6)$ are identified with the fundamental and anti-fundamental representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$, respectively. As a result $Spin(6) = SU(4)$ has one type of a non-trivial orbit in each of these two representations which is a 7-sphere and has isotropy group $SU(3)$. Therefore assuming that ϵ is chiral or anti-chiral, it can be put in any direction in the $\mathbf{4}$ or $\bar{\mathbf{4}}$ representation, respectively.

To solve the KSE, it is convenient to choose a “simple” representative for the Killing spinor. To do this assume that ϵ is chiral and observe that

$$\Lambda^{\text{ev}}(\mathbb{C}^3) = \mathbb{C}\langle 1, e_{ij} \rangle , \quad (2.3)$$

where $e_{ij} = e_i \wedge e_j$. As ϵ can be put in any direction, one can choose without loss of generality that $\epsilon = 1$. Then the KSE (2.1) can be rewritten as

$$F_{\mu\nu}\Gamma^{\mu\nu}1 = 0 . \quad (2.4)$$

To find the linear system associated to the above equation, introduce a Hermitian basis in the space of gamma matrices as

$$\Gamma_\alpha = \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+3}) , \quad \Gamma_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+3}) , \quad (2.5)$$

where now

$$\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha = 0 , \quad \Gamma_{\bar{\alpha}}\Gamma_{\bar{\beta}} + \Gamma_{\bar{\beta}}\Gamma_{\bar{\alpha}} = 0 , \quad \Gamma_\alpha\Gamma_{\bar{\beta}} + \Gamma_{\bar{\beta}}\Gamma_\alpha = 2\delta_{\alpha\bar{\beta}} . \quad (2.6)$$

Expanding (2.4) in this Hermitian basis, one finds the linear system

$$F_{\bar{\alpha}\bar{\beta}}\Gamma^{\bar{\alpha}\bar{\beta}}1 + 2\delta^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}}1 = 0 . \quad (2.7)$$

Since $(1, \Gamma^{\bar{\alpha}\bar{\beta}}1)$ for $\bar{\alpha} < \bar{\beta}$ is a basis in $\Lambda^{\text{ev}}(\mathbb{C}^3)$, we conclude that the solution to the linear system is

$$F_{\bar{\alpha}\bar{\beta}} = 0 , \quad \delta^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = 0 . \quad (2.8)$$

To interpret these conditions, one can define a 2-form spinor bilinear as

$$\omega = \frac{i}{2}\langle 1, \Gamma_{ij}1 \rangle dx^i \wedge dx^j , \quad (2.9)$$

where $\langle \cdot, \cdot \rangle$ is the Dirac spinor inner product which is the same as the Hermitian inner product on the space of spinors $\Lambda^*(\mathbb{C}^3)$ induced from that on \mathbb{C}^3 . This is a Hermitian form and together with the metric on \mathbb{R}^6 give rise to a complex structure on \mathbb{R}^6 . This is the complex structure I which is invariant under the isotropy group $SU(3)$ of the Killing spinor. Therefore in complex coordinates on \mathbb{R}^6 with respect to I

$$ds^2 = \delta_{ij}dx^i dx^j = 2\delta_{\alpha\bar{\beta}} dz^\alpha dz^{\bar{\beta}} , \quad \omega = -i\delta_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} . \quad (2.10)$$

Then the above conditions (2.8) imply that F is a (2,0) and (1,1) form with respect to I , and the trace of the (1,1) component vanishes. Of course if F is real, then the (2,0) component vanishes as well as it is the complex conjugate of (0,2) component. These conditions can immediately be recognized as instanton equations on 6-dimensions refereed to as the Hermitian-Einstein conditions on the gauge fields.

2.4 Spinorial geometry and supergravity

The KSEs of supergravity theories are the vanishing conditions of the supersymmetry variations of the fermions of the theory evaluated at the locus where all fermions vanish. The unknowns are the supersymmetry parameters which are taken to be commuting spinors. The KSEs of supergravity theories are separated into parallel transport equations associated with the supersymmetry variations of the gravitini, and algebraic equations associated with the supersymmetry variations of the remaining fermions of the theory. Schematically, they are written as

$$\begin{aligned}\mathcal{D}_\mu \epsilon &\equiv \nabla_\mu \epsilon + \Sigma_\mu(g, F) \epsilon = 0 , \\ \mathcal{A} \epsilon &\equiv \mathcal{A}(g, F) \epsilon = 0 ,\end{aligned}\tag{2.11}$$

where ϵ is the supersymmetry parameter. The covariant derivative, \mathcal{D} , often called the supercovariant derivative of the supergravity theory, begins with the spin connection of the spacetime metric, ∇ , and receives a correction Σ which depends on the metric and the remaining bosonic fields of the theory. Σ typically contains terms of higher order than quadratic in a skew-symmetric product expansion of gamma matrices. On the other hand \mathcal{A} is an algebraic equation on ϵ which depends on the bosonic fields of the theory.

The gauge transformations of KSEs are those transformations which leave the form of the KSEs covariant. The gauge group of the KSEs of a supergravity theory includes the spin group of the spacetime, and in gauged supergravity also includes the gauge group of the theory. The holonomy group of the supercovariant connection \mathcal{D} of a generic background includes the gauge group of the KSEs and but in many supergravity theories is a much larger group.

In the current context, a solution of the KSEs means to specify the differential geometric conditions on the bosonic fields of the supergravity theory such that the KSEs admit an $\epsilon \neq 0$ as a solution. The number N of linearly independent solutions ϵ , called Killing spinors, of the KSEs is the number of supersymmetries preserved by the background. To find a supersymmetric solution in addition to solving the KSEs, one also has to solve the field equations of the theory. Typically the KSEs imply some of the field equations but not necessarily all.

Spinorial geometry utilizes the gauge group of the KSEs of a supergravity theory to choose the Killing spinors. Then as in the gauge theory paradigm, the KSEs turn into a linear system which is solved to express some of the fields in terms of the geometry and also identify the conditions on the geometry required for the KSEs to admit a solution. The geometric conditions are typically expressed as a linear relation between the components of the spin connection ∇ .

3 $(1, 0)$ supergravity in six dimensions

The main task is to describe the solution of the KSEs of 6-dimensional $(1,0)$ supergravity coupled to any number of tensor, vector and scalar multiplets presented in [20]. Solutions of the KSEs of 6-dimensional supergravities in special cases have been investigated before in [40, 41, 42, 43, 44].

3.1 Fields and KSEs

Supergravity in six dimensions [28, 29, 31] with (1,0) supersymmetry, 8 real supercharges, is constructed from four different supersymmetry multiplets the following. The gravitational multiplet which has field content a graviton g , an anti-self-dual 2-form gauge potential B and a gravitino Ψ . The tensor multiple which consists of a self-dual 2-form gauge potential b , a scalar ϕ and fermion χ which has chirality opposite to that of the gravitino. The vector or gauge multiplet which has a vector gauge potential A and a fermion λ with the same chirality as that of gravitino, and a scalar or hyper-multiplet which consists of four real scalars q and a fermion ψ which has opposite chirality to that of gravitino. A mnemonic of the field content of the multiplets is

$$\begin{aligned} \text{gravity multiplet} & : g_{\mu\nu} , \quad B_{\mu\nu} ; \quad \Psi_\mu \\ \text{tensor multiplet} & : b_{\mu\nu} , \quad \phi ; \quad \chi \\ \text{vector multiplet} & : A_\mu ; \quad \lambda \\ \text{scalar multiplet} & : q ; \quad \psi \end{aligned} \tag{3.1}$$

The system that we consider is (1,0) supergravity coupled to n_T tensor, n_V vector and n_H scalar multiplets. All the fermions of the four multiplets are chiral and satisfy the symplectic-Majorana spinor condition. The symplectic-Majorana condition is a reality condition which is imposed on the complex chiral spinors of $Spin(5, 1)$. This condition utilizes the invariant $Sp(1)$ and $Sp(n_H)$ forms to impose a reality condition on the complex spinors preserving chirality. Suppose that the Dirac or Weyl spinors λ and χ transform under the fundamental representations of $Sp(1)$ and $Sp(n_H)$, respectively. The symplectic Majorana condition is given by

$$\lambda^i = \epsilon^{ij} C \bar{\lambda}_j^T , \quad \chi^a = \epsilon^{ab} C \bar{\chi}_b^T , \tag{3.2}$$

where C is the charge conjugation matrix and ϵ^{ij} and ϵ^{ab} are the symplectic invariant forms of $Sp(1)$ and $Sp(n_H)$, respectively, and $i, j = 1, 2$ and $a, b = 1, \dots, 2n_H$.

To describe the KSEs of (1,0) supergravity coupled to tensor, vector and scalar multiplets, we use a formulation¹ proposed by [31]. The theory has $n_T + 1$ 2-form gauge potentials $B^{\underline{r}}$, $\underline{r} = 0, 1, \dots, n_T$. One of the 2-form potentials is associated with the gravitational multiplet and the remaining n_T with the tensor multiplets. Let us denote the corresponding 3-form field strengths with $G^{\underline{r}}$. To continue, the scalar fields of the tensor multiplets parameterize the coset space $SO(1, n_T)/SO(n_T)$. A convenient way to describe this coset space is to choose a local section S as

$$S = \begin{pmatrix} v_{\underline{r}} \\ x_{\underline{r}}^I \end{pmatrix} , \quad I = 1, \dots, n_T \tag{3.3}$$

Since $S \in SO(1, n_T)$, one has $\tilde{S}\eta S = \eta$ where η is the Lorentz metric in $n_T + 1$ -dimensions. In particular

$$v_{\underline{r}} v^{\underline{r}} = 1 , \quad v_{\underline{r}} v_{\underline{s}} - \sum_I x_{\underline{r}}^I x_{\underline{s}}^I = \eta_{\underline{r}\underline{s}} , \quad v^{\underline{r}} x_{\underline{r}}^I = 0 . \tag{3.4}$$

¹We use a different normalization for some of the fields from that in [31]. Our normalization is similar to that of heterotic supergravity.

The scalars of the hypermultiplet parameterize a Quaternionic Kähler manifold \mathcal{Q} . This is a Riemannian manifold equipped with a quaternionic structure, ie endomorphisms I_τ , $\tau = 1, 2, 3$, of the tangent bundle such that $I_{\tau_1} I_{\tau_2} = -\delta_{\tau_1 \tau_2} \mathbf{1} + \epsilon_{\tau_1 \tau_2 \tau_3} I_{\tau_3}$, and whose Levi-Civita connection has holonomy $Sp(n_H) \cdot Sp(1)$, see [30] for a mathematical description. Such a manifold admits a frame E such that the metric and the endomorphisms can be written as

$$g_{MN} = E_M^{\text{ai}} E_N^{\text{bj}} \epsilon_{\text{ab}} \epsilon_{\text{ij}} , \quad (I_\tau)^M{}_N = -i(\sigma_\tau)^i{}_j \delta^{\text{a}}_{\text{b}} E_{\text{ai}}^M E_N^{\text{bj}} , \quad (3.5)$$

where ϵ_{ab} and ϵ_{ij} are the invariant $Sp(n_H)$ and $Sp(1)$ 2-forms, respectively, and σ_τ are the Pauli matrices. The spin connection, which has holonomy $Sp(n_H) \cdot Sp(1)$, decomposes as $\mathcal{A}_M = (\mathcal{A}_{M\text{b}}, \mathcal{A}_{Mj}^i)$.

In [31] to include vector multiplets with (non-abelian) gauge potential A_μ^{m} , one assumes that the Quaternionic Kähler manifold² \mathcal{Q} of the hypermultiplet is $Sp(n_H, 1)/Sp(1) \times Sp(n_H)$ and gauges the maximal compact isometry subgroup $Sp(1) \times Sp(n_H)$. So the gauge group of the theory is $H = Sp(1) \times Sp(n_H) \times K$, where K is a product of semi-simple groups which does not act on the scalars. Let ξ_{m_1} and ξ_{m_2} be the vector fields generated on $Sp(n_H, 1)/Sp(1) \times Sp(n_H)$ by the action of $Sp(1)$ and $Sp(n_H)$, respectively. Under these assumptions, one defines

$$\begin{aligned} H_{\mu\nu\rho} &= v_{\underline{x}} G_{\mu\nu\rho}^x , \quad H_{\mu\nu\rho}^I = x_{\underline{x}}^I G_{\mu\nu\rho}^x , \quad \mathcal{C}_{\mu j}^i = D_\mu q^M \mathcal{A}_{Mj}^i , \\ T_\mu^I &= x_{\underline{x}}^I \partial_\mu v^x , \quad V_\mu^{\text{ai}} = E_M^{\text{ai}} D_\mu q^M , \quad F_{\mu\nu}^{\text{m}} = \partial_\mu A_\nu^{\text{m}} - \partial_\nu A_\mu^{\text{m}} + f_{\text{np}}^{\text{m}} A_\mu^{\text{n}} A_\nu^{\text{p}} , \\ (\mu^{\text{m}_1})^i{}_j &= -\frac{2}{v_{\underline{x}} c^{\text{x}1}} \mathcal{A}_{Mj}^i \xi^{M\text{m}_1} , \quad (\mu^{\text{m}_2})^i{}_j = -\frac{2}{v_{\underline{x}} c^{\text{x}2}} \mathcal{A}_{Mj}^i \xi^{M\text{m}_2} , \quad (\mu^{\text{m}_3})^i{}_j = 0 , \end{aligned} \quad (3.6)$$

where the gauge index m_3 ranges over the gauge subgroup K , q^M are the scalars of the hypermultiplet,

$$\nabla_\mu \epsilon^i = \partial_\mu \epsilon^i + \frac{1}{4} \Omega_{\mu, mn} \gamma^{mn} \epsilon^i , \quad D_\mu q^M = \partial_\mu q^M - A_\mu^{\text{m}} \xi_{\text{m}}^M , \quad (3.7)$$

and Ω is the frame connection of spacetime. It is understood that $\xi_{\text{m}_3} = 0$ as K does not act on the scalars of the hypermultiplet. Clearly F^{m} are the field strengths of the gauge potentials A^{m} and f are the structure constants of the gauge group H . We refer to μ 's as the moment maps, see [39].

It remains to define the field strengths G^x . These are given by

$$G_{\mu\nu\rho}^x = 3\partial_{[\mu} B_{\nu\rho]}^x + c^{\text{x}1} CS(A^{Sp(1)})_{\mu\nu\rho} + c^{\text{x}2} CS(A^{Sp(n_H)})_{\mu\nu\rho} + c^{\text{x}K} CS(A^K)_{\mu\nu\rho} , \quad (3.8)$$

where c^x 's are constants, one for each copy of the gauge group, and $CS(A)$'s are the Chern-Simons 3-forms. Observe that the constants $c^{\text{x}1}$ and $c^{\text{x}2}$ enter in the definition of μ 's in (3.6).

The duality condition on G is given by

$$\zeta_{\underline{x}\underline{s}} G_{\mu_1\mu_2\mu_3}^{\text{s}} = \frac{1}{3!} \epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2\nu_3} G_{\underline{x}\nu_1\nu_2\nu_3} , \quad (3.9)$$

²It is likely that this assumption is not necessary and a more general class of models can exist. Moreover μ may be related to moment maps [39] of Quaternionic Kähler geometry.

where

$$\zeta_{\underline{rs}} = v_{\underline{r}} v_{\underline{s}} + \sum_I x_{\underline{r}}^I x_{\underline{s}}^I . \quad (3.10)$$

Note that the duality conditions for H and H^I are opposite. In our conventions, H is anti-self-dual while H^I are self-dual.

The Lagrangian of the theory is

$$\begin{aligned} e^{-1} \mathcal{L} = & -\frac{1}{4} R + \frac{1}{48} \zeta_{rs} G_{\mu\nu\rho}^r G^{s\ \mu\nu\rho} - \frac{1}{4} \partial_\mu v^r \partial^\mu v_r + \frac{1}{8} v_r c^r F_{\mu\nu}^m F^{m\mu\nu} \\ & - \frac{1}{64e} \epsilon^{\mu\nu\rho\sigma\delta\tau} B_{\mu\nu}^r c_r F_{\rho\sigma}^m F_{\delta\tau}^m + \frac{1}{2} g_{MN} D_\mu q^M D^\mu q^N \\ & - \frac{1}{2v_r c^r} \mathcal{A}_{M\ i}^i \mathcal{A}_{N\ j}^i \xi^{mM} \xi^{mN} . \end{aligned} \quad (3.11)$$

It is understood that to derive the field equations one first varies 3-form field strengths and then imposes self-duality and anti-self-duality conditions.

The supersymmetry transformations of (1,0) supergravity fermions coupled to n_T tensor, n_V vector and n_H scalar multiplets evaluated at the locus where all the fermion fields vanish are

$$\begin{aligned} \delta\Psi_\mu^i &= \nabla_\mu \epsilon^i - \frac{1}{8} H_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon^i + \mathcal{C}_\mu^i{}_j \epsilon^j , \\ \delta\lambda^{mi} &= -\frac{1}{2\sqrt{2}} F_{\mu\nu}^m \gamma^{\mu\nu} \epsilon^i - \frac{1}{\sqrt{2}} (\mu^m)^i{}_j \epsilon^j , \\ \delta\chi^{Ii} &= \frac{i}{2} T_\mu^I \gamma^\mu \epsilon^i - \frac{i}{24} H_{\mu\nu\rho}^I \gamma^{\mu\nu\rho} \epsilon^i , \\ \delta\psi^a &= i\gamma^\mu \epsilon_i V_\mu^{ai} , \end{aligned} \quad (3.12)$$

where the fermions are defined as in (3.1). The KSEs of the (1,0) supergravity are derived from setting all the above transformations to zero and they will be referred to as gravitino, gaugini, tensorini and hyperini KSEs, respectively. Although to write the above KSEs we have used the particular supergravity theory described in [31], the form of these transformations is model independent. The reason is that these transformations are the most general supersymmetry transformations that one can write. So although the expression of the field strengths in terms of the gauge potentials will change from model to model depending on the details of the couplings, the actual form of the transformations does not. The application of the spinorial geometry method to solve the KSEs does not depend of the details on how the field strengths depend on the physical fields. As a results it applies to all (1,0) supersymmetric models and not only to the one described in this section.

3.2 A realization of spinors in terms of forms

The most effective way to represent the spinors of (1,0) supergravity in terms of forms is to identify the symplectic Majorana-Weyl spinors of $Spin(5,1)$ with the $SU(2)$ invariant Majorana-Weyl spinors of $Spin(9,1)$ [18, 20]. To do this explicitly, the Dirac spinors of $Spin(9,1)$ are identified with $\Lambda^*(\mathbb{C}^5)$, and the positive and negative chirality spinors are

the even and odd degree forms, respectively. A realization of the gamma matrices of $\text{Clif}(\mathbb{R}^{9,1})$ is given by

$$\begin{aligned}\Gamma_0 &= -e_5 \wedge + e_5 \lrcorner, & \Gamma_5 &= e_5 \wedge + e_5 \lrcorner, \\ \Gamma_i &= e_i \wedge + e_i \lrcorner, & \Gamma_{i+5} &= i(e_i \wedge - e_i \lrcorner), \quad i = 1, 2, 3, 4,\end{aligned}\quad (3.13)$$

where e_i , $i = 1, \dots, 5$, is a Hermitian basis in \mathbb{C}^5 . The gamma matrices of $\text{Clif}(\mathbb{R}^{5,1})$ are identified as

$$\gamma_\mu = \Gamma_\mu, \quad \mu = 0, 1, 2; \quad \gamma_\mu = \Gamma_{\mu+2}, \quad \mu = 3, 4, 5. \quad (3.14)$$

Therefore the positive chirality Weyl spinors of $\text{Spin}(5, 1) = SL(2, \mathbb{H})$ are $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5 \rangle) = \mathbb{H}^2$. The symplectic Majorana-Weyl condition of $\text{Spin}(5, 1)$ is the Majorana-Weyl condition of $\text{Spin}(9, 1)$ spinors, ie

$$\epsilon^* = \Gamma_{67}\Gamma_{89}\epsilon, \quad (3.15)$$

where $\epsilon \in \Lambda^{\text{ev}}\mathbb{C}\langle e_1, e_2, e_5 \rangle \otimes \Lambda^*\mathbb{C}\langle e_{34} \rangle$. In particular a basis for the symplectic Majorana-Weyl spinors is

$$\begin{aligned}1 + e_{1234}, & \quad i(1 - e_{1234}), \quad e_{12} - e_{34}, \quad i(e_{12} + e_{34}), \\ e_{15} + e_{2534}, & \quad i(e_{15} - e_{2534}), \quad e_{25} - e_{1534}, \quad i(e_{25} + e_{1534}).\end{aligned}\quad (3.16)$$

Observe that the above basis selects the diagonal of two copies of the Weyl representation of $\text{Spin}(5, 1)$, where the first copy is $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5 \rangle)$ while the second copy is $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5 \rangle) \otimes \mathbb{C}\langle e_{34} \rangle$. The $SU(2)$ acting on the auxiliary directions e_3 and e_4 leaves the basis invariant.

The KSEs of 6-dimensional supergravity can be rewritten in terms of the 10-dimensional notation we have introduced above. For this, we define $\rho^{r'}$, $r' = 1, 2, 3$, such that

$$\rho^1 = \frac{1}{2}(\Gamma_{38} + \Gamma_{49}), \quad \rho^2 = \frac{1}{2}(\Gamma_{89} - \Gamma_{34}), \quad \rho^3 = \frac{1}{2}(\Gamma_{39} - \Gamma_{48}). \quad (3.17)$$

Observe that these are the generators of the Lie algebra $Sp(1)$ as it acts on the basis (3.16). Using this the KSEs can be rewritten as

$$\begin{aligned}\mathcal{D}_\mu \epsilon &\equiv \left(\nabla_\mu - \frac{1}{8} H_{\mu\nu\rho} \gamma^{\nu\rho} + \mathcal{C}_\mu^{r'} \rho_{r'} \right) \epsilon = 0, \\ \left(\frac{1}{4} F_{\mu\nu}^{\text{m}} \gamma^{\mu\nu} + \frac{1}{2} \mu_{r'}^{\text{m}} \rho^{r'} \right) \epsilon &= 0, \\ \left(\frac{i}{2} T_\mu^I \gamma^\mu - \frac{i}{24} H_{\mu\nu\rho}^I \gamma^{\mu\nu\rho} \right) \epsilon &= 0, \\ i\gamma^\mu \epsilon_i V_\mu^{\text{ai}} &= 0.\end{aligned}\quad (3.18)$$

In the hyperini KSE, it is understood that

$$\epsilon_1 = -\epsilon^2, \quad \epsilon_2 = \Gamma_{34}\epsilon^1, \quad (3.19)$$

where ϵ^1 and ϵ^2 are the components of ϵ in the two copies of the Weyl representation used to construct the symplectic Majorana-Weyl representation as explained below (3.16).

4 Solution of KSEs

To solve the KSEs of a supergravity theory, it is customary to begin with the gravitino KSE. This is because it is a parallel transport equation and so has a significant role in the description of the geometry of spacetime. As a result, we shall present a detailed analysis of the solutions of gravitino KSE of (1,0) supergravity. The solution of the remaining KSEs will be presented in some detail for backgrounds preserving one supersymmetry. For the rest of the cases, only a brief summary will be given. The omitted details and the proof of the statements we have used to solve all KSEs can be found in the original paper [20].

4.1 Gravitino KSE

To solve KSEs in the context of spinorial geometry, the main task is to find the representatives of the Killing spinors up to gauge transformations. The gauge group of the KSEs of (1,0) supergravity is $Spin(5,1) \cdot Sp(1)$. This is the same as the (reduced) holonomy group of the supercovariant connection \mathcal{D} in (3.18) for a generic background. To see the latter, the curvature \mathcal{R} of the supercovariant connection is

$$\mathcal{R}_{\mu\nu} \equiv [\mathcal{D}_\mu, \mathcal{D}_\nu] = \frac{1}{4} \hat{R}_{\mu\nu, \rho\sigma} \gamma^{\rho\sigma} + \mathcal{F}_{\mu\nu}^{r'} \rho_{r'} , \quad (4.1)$$

where

$$\mathcal{F}_{\mu\nu}^{r'} = \partial_\mu C_\nu^{r'} - \partial_\nu C_\mu^{r'} + 2\epsilon^{r'}_{s't'} C_\mu^{s'} C_\nu^{t'} , \quad (4.2)$$

and \hat{R} is the curvature of the connection, $\hat{\nabla}$, with skew-symmetric torsion H defined as

$$\hat{\nabla}_\mu Y^\nu = \nabla_\mu Y^\nu + \frac{1}{2} H^\nu_{\mu\lambda} Y^\lambda . \quad (4.3)$$

For any two vectors X, Y of spacetime, $\mathcal{R}(X, Y)$ spans a $\mathfrak{spin}(5,1) \oplus \mathfrak{sp}(1)$ algebra and so the holonomy of \mathcal{D} is contained in $Spin(5,1) \cdot Sp(1)$.

Now the solutions $\epsilon \neq 0$ of the gravitino KSE, $\mathcal{D}_\mu \epsilon = 0$, must satisfy $\mathcal{R}\epsilon = 0$. Thus either the Killing spinors ϵ have a trivial isotropy group in the generic holonomy group $Spin(5) \cdot Sp(1)$ in which case

$$\hat{R} = 0 , \quad \mathcal{F} = 0 . \quad (4.4)$$

and so the spacetime is parallelizable with respect to a connection with skew-symmetric torsion, or they have a non-trivial isotropy group in the generic holonomy group $Spin(5) \cdot Sp(1)$. In the former case, all such spacetimes are locally isometric to group manifolds with anti-self-dual structure constants. In the latter case, the holonomy of the supercovariant \mathcal{D} connection reduces to that of the isotropy group of the Killing spinors. So to complete the solution of the gravitino KSE, the subgroups of $Spin(5,1) \cdot Sp(1)$ which leave spinors invariant must be identified.

4.1.1 Non-trivial isotropy groups

To find the isotropy groups of spinors, it is known that the action of $Spin(5, 1) \cdot Sp(1)$ on the space of symplectic Majorana-Weyl spinors can be described in terms of quaternions. In particular, the chiral symplectic Majorana spinors are identified with \mathbb{H}^2 and $Spin(5, 1)$ with $SL(2, \mathbb{H})$, $Spin(5, 1) = SL(2, \mathbb{H})$. Then $Spin(5, 1) \cdot Sp(1)$ acts on \mathbb{H}^2 as

$$(A, a)\mathbf{v} = A\mathbf{v}\bar{a} \quad (4.5)$$

where $(A, a) \in Spin(5, 1) \cdot Sp(1)$, $\mathbf{v} \in \mathbb{H}^2$ and A acts with a quaternionic matrix multiplication, and where \bar{a} is the quaternionic conjugate of a , $a\bar{a} = 1$. Using, this it is easy to see that there is a single non-trivial orbit of $Spin(5, 1) \cdot Sp(1)$ on the symplectic Majorana-Weyl spinors with isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$. To continue, we have to determine the action of $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ on \mathbb{H}^2 . Decomposing $\mathbb{H}^2 = \mathbb{R} \oplus \text{Im}\mathbb{H} \oplus \mathbb{H}$, where \mathbb{R} is chosen to be along the first invariant spinor, the action of the isotropy group is

$$\text{Im}\mathbb{H} \oplus \mathbb{H} \rightarrow a\text{Im}\mathbb{H}\bar{a} \oplus b\mathbb{H}\bar{a} , \quad (4.6)$$

where $(a, b) \in Sp(1) \cdot Sp(1)$. There are two possibilities. Either the second invariant spinor lies in $\text{Im}\mathbb{H}$ or in \mathbb{H} . It cannot lie in both because if there is a non-trivial component in \mathbb{H} , there is a \mathbb{H} transformation in $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ such that the component in $\text{Im}\mathbb{H}$ can be set to zero. Now if the second spinor lies in $\text{Im}\mathbb{H}$, the isotropy group is $Sp(1) \cdot U(1) \ltimes \mathbb{H}$. On the other hand if it lies in \mathbb{H} , the isotropy group is $Sp(1)$. This concludes the analysis for two invariant spinors.

There is no subgroup in $Spin(5, 1) \cdot Sp(1)$ which leaves invariant strictly 3 spinors. For 4 invariant spinors, there are two cases to consider. Either all four invariant spinors span the first copy of \mathbb{H} in \mathbb{H}^2 and the isotropy group is $Sp(1) \ltimes \mathbb{H}$, or 2 lie in the first copy and the other 2 lie in the second copy of \mathbb{H} in \mathbb{H}^2 and the isotropy group is $U(1)$. The isotropy group of more than 4 linearly independent spinors is $\{1\}$.

It remains to find representatives of the solutions to the gravitino KSE up to gauge transformations. Observe that the generic holonomy group and the gauge group of the KSEs coincide and both act in the same way on the symplectic Majorana-Weyl spinors. Repeating the analysis we have done to identify the isotropy group of spinors in $Spin(5, 1) \cdot Sp(1)$, it is straightforward to find the representatives of the invariant spinors. For example in the case of one invariant spinor, since $Spin(5, 1) \cdot Sp(1)$ acts on \mathbb{H}^2 with one non-trivial orbit which is dense, any spinor can be chosen as a representative. Moreover the representatives can be expressed as forms using the description of spinors as in section 3.2 The isotropy groups of spinors in $Spin(5, 1) \cdot Sp(1)$ as well as representatives of the invariant spinors have been summarized in table 1.

4.2 Solution of remaining KSEs

One expects that given some parallel spinors, ie a solution of the gravitino KSE, only some of them will be Killing, ie only some will also solve the remaining KSEs. Therefore to find all supersymmetric backgrounds, one has to investigate which of the parallel spinors also solve the remaining KSEs. There are many possibilities and the analysis is rather involved. Because of this, it will not be presented here and can be found in [20]. However

N	Isotropy Groups	Spinors
1	$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}$
2	$(Sp(1) \cdot U(1)) \ltimes \mathbb{H}$	$1 + e_{1234} , i(1 - e_{1234})$
4	$Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234} , i(1 - e_{1234}) , e_{12} - e_{34} , i(e_{12} + e_{34})$
2	$Sp(1)$	$1 + e_{1234} , e_{15} + e_{2345}$
4	$U(1)$	$1 + e_{1234} , i(1 - e_{1234}) , e_{15} + e_{2345} , i(e_{15} - e_{2345})$

Table 1: The first column gives the number of invariant spinors, the second column the associated isotropy groups and the third the representatives of the invariant spinors. Observe that if three spinors are invariant, then there is a fourth one. Moreover the isotropy group of more than 4 spinors is the identity.

the final result is rather straightforward. Apart from one case that has to do with the hyperini KSE, to identify all supersymmetric backgrounds suffices to consider the cases where all parallel spinors also solve the remaining KSEs and so are Killing. The results are summarized in table 2

$\text{hol}(\mathcal{D})$	N
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	1
$Sp(1) \cdot U(1) \ltimes \mathbb{H}$	*, 2
$Sp(1) \ltimes \mathbb{H}$	*, *, 3, 4
$Sp(1)$	*, 2
$U(1)$	*, *, -, 4
$\{1\}$	*,*,*, *,-, -, -, 8

Table 2: In the columns are the holonomy groups that arise from the solution of the gravitino KSE and the number N of supersymmetries, respectively. * entries denote the cases that occur but are special cases of others with the same number of supersymmetries but with less parallel spinors. The - entries denote cases which do not occur. The Killing spinors for $N = 1, 2, 4$ are the same as those given in table 1 while for $N = 3$ the Killing spinors are given in (4.7).

To complete the analysis, it suffices to give the Killing spinors of the $N = 3$ case; all the remaining ones can be found in table 1. The three Killing spinors can be chosen as

$$1 + e_{1234} , i(1 - e_{1234}) , e_{12} - e_{34} . \quad (4.7)$$

It turns out that if the gravitino, tensorini and gaugini KSEs admit (4.7) as a solution, then they admit also $i(e_{12} + e_{34})$ as a solution. Thus all the parallel spinors of this case solve the three out of four KSEs. However, this is not the case for the hyperini KSE. The conditions that arise on evaluating the hyperini KSE on (4.7) are different from those that one finds when the same KSE is evaluated on all 4 $Sp(1) \ltimes \mathbb{H}$ -invariant spinors. As a result, there is a distinct case preserving strictly 3 supersymmetries.

5 Geometry

Having found representatives for the Killing spinors, it is straightforward to evaluate the KSEs and derive the linear systems for all cases. The linear systems can then be solved to derive the conditions required on the fields so that the KSEs admit a solution. The analysis is similar to the paradigm in section 2. Before, we proceed with a case by case analysis, it is instructive to first observe that in all cases the solution of the gravitino KSE can be summarized by stating that the holonomy of the supercovariant connection is included in the isotropy group G of the parallel spinors, ie

$$\text{hol}(\mathcal{D}) \subseteq G , \quad (5.1)$$

where all groups G are presented in table 1. There are several ways that this condition can be expressed in a differential geometric way. One is to consider the forms constructed as Killing spinor bilinears. Given two spinors ϵ_1 and ϵ_2 , one class of form bilinears is

$$\tau = \frac{1}{k!} B(\epsilon_1, \gamma_{\mu_1 \dots \mu_k} \epsilon_2) e^{\mu_1} \wedge \dots \wedge e^{\mu_k} , \quad (5.2)$$

where $B(\epsilon_1, \epsilon_2) = \langle \Gamma_{06789} \epsilon_1^*, \epsilon_2 \rangle$ is the Majorana inner product in the basis chosen in section 3.2, and where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $\Lambda(\mathbb{C}^5)$. Assuming that ϵ_1 and ϵ_2 satisfy the gravitino KSE, it is easy to see that

$$\hat{\nabla}_\nu \tau = 0 . \quad (5.3)$$

The form τ is covariantly constant with respect to $\hat{\nabla}$ - the $Sp(1)$ connection $\mathcal{C}^{r'}$ does not contribute in the covariant constancy condition.

Another class of bilinears is the $\mathfrak{sp}(1)$ -valued forms

$$\tau^{r'} = \frac{1}{k!} B(\epsilon_1, \gamma_{\mu_1 \dots \mu_k} \rho^{r'} \epsilon_2) e^{\mu_1} \wedge \dots \wedge e^{\mu_k} . \quad (5.4)$$

Assuming again that ϵ_1 and ϵ_2 satisfy the gravitino KSE, one finds that

$$\hat{\nabla}_\nu \tau^{r'} + 2 \mathcal{C}_\nu^{s'} \epsilon^{r'}_{s't'} \tau^{t'} = 0 . \quad (5.5)$$

Observe that the $\mathfrak{sp}(1)$ -valued form bi-linears are twisted with respect to the $Sp(1)$ connection $\mathcal{C}^{r'}$. So $\tau^{r'}$ are not forms but rather vector bundle valued forms. However for simplicity in what follows, we shall refer to both τ and $\tau^{r'}$ as forms.

To solve the gravitino and identify the conditions on the geometry of spacetime, we shall investigate the consequences of (5.3) and (5.5) in each case. Then we shall investigate the conditions on the fields imposed by the remaining KSEs.

5.1 N=1

5.1.1 Gravitino KSE and Spacetime geometry

To express the form spinor bilinears for backgrounds preserving one supersymmetry, it is convenient to introduce a lightcone-Hermitian frame on the spacetime, $(e^-, e^+, e^\alpha, e^{\bar{\alpha}})$, $\alpha = 1, 2$, ie the metric is written as

$$ds^2 = 2e^+ e^- + \delta_{ij} e^i e^j = 2(e^+ e^- + \delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}}) . \quad (5.6)$$

This frame can be chosen such that the form spinor bilinears are

$$e^-, \quad e^- \wedge \omega^1, \quad e^- \wedge \omega^2, \quad e^- \wedge \omega^3, \quad (5.7)$$

where e^- is a null one-form and

$$\omega^1 = -i\delta_{\alpha\bar{\beta}}e^\alpha \wedge e^{\bar{\beta}}, \quad \omega^2 = -e^1 \wedge e^2 - e^{\bar{1}} \wedge e^{\bar{2}}, \quad \omega^3 = i(e^1 \wedge e^2 - e^{\bar{1}} \wedge e^{\bar{2}}). \quad (5.8)$$

Clearly $\omega^{r'}$, $r' = 1, 2, 3$ are Hermitian forms for a quaternionic structure J_1, J_2, J_3 , $J_{r'}J_{s'} = -\delta_{r's'}\mathbf{1} + \epsilon_{r's't'}J_{t'}$, on the directions transverse to (e^+, e^-) .

The conditions that the gravitino KSE imposes on the spacetime geometry can be rewritten as

$$\hat{\nabla}_\mu e^- = 0, \quad \hat{\nabla}_\mu(e^- \wedge \omega^{r'}) + 2\mathcal{C}_\mu^{s'}\epsilon^{r'}_{s't'}(e^- \wedge \omega^{t'}) = 0. \quad (5.9)$$

The second equation can be thought as the Lorentzian analogue of the Quaternionic Kähler with torsion condition of [45]. The integrability conditions to these parallel transport equations are

$$\hat{R}_{\mu_1\mu_2,+\nu} = 0, \quad -\hat{R}_{\mu_1\mu_2,}^k \omega^{r'}_{kj} + (j, i) + 2\mathcal{F}_{\mu_1\mu_2}^{s'}\epsilon^{r'}_{s't'}\omega_{ij}^{t'} = 0. \quad (5.10)$$

In addition to this, the torsion H has to be anti-self-dual in 6 dimensions. The conditions for this in the lightcone-Hermitian frame can be written as as

$$H_{+\alpha\beta} = H_{+\alpha}{}^\alpha = 0, \quad H_{-+\bar{\alpha}} + H_{\bar{\alpha}\beta}{}^\beta = 0, \quad H_{-1\bar{1}} - H_{-2\bar{2}} = 0, \quad H_{-1\bar{2}} = 0, \quad (5.11)$$

where $\epsilon_{-+1\bar{1}2\bar{2}} = \epsilon_{013245} = -1$. Notice that from the 4-dimensional perspective of directions transverse to (e^+, e^-) , H_{+ij} is an anti-self-dual while H_{-ij} is a self-dual 2-form, respectively.

To specify the spacetime geometry, one has to solve (5.9) subject to (5.11). The first condition in (5.9) implies that

$$\mathcal{L}_X g = 0, \quad de^- = i_X H. \quad (5.12)$$

ie that the vector field X dual to 1-form e^- is *Killing* and the $i_X H$ component of H is given by the exterior derivative of the bilinear e^- . In fact, X leaves invariant all the fields of the theory. From this, it is easy to see that the torsion 3-form can be written as

$$H = e^+ \wedge de^- + \frac{1}{2}H_{-ij}e^- \wedge e^i \wedge e^j + \tilde{H}, \quad \tilde{H} = \frac{1}{3!}\tilde{H}_{ijk}e^i \wedge e^j \wedge e^k. \quad (5.13)$$

Anti-self-duality of H relates the \tilde{H} component to de^- . In particular, one has that

$$\tilde{H} = -\frac{1}{3!}(de^-)_{-\ell} \epsilon^\ell_{ijk} e^i \wedge e^j \wedge e^k. \quad (5.14)$$

This solves the first condition in (5.9). To solve the remaining three conditions, consider first the parallel transport equation in (5.9) along the light-cone directions. Since H_{+ij} is anti-self-dual, one has that

$$\mathcal{D}_+\omega^{r'} = \nabla_+\omega^{r'} + 2\mathcal{C}_+^{s'}\epsilon^{r'}_{s't'}\omega^{t'} = 0. \quad (5.15)$$

As we shall see from the hyperini KSE (5.30), $\mathcal{C}_+ = 0$, and so the above condition becomes a restriction on the geometry

$$\nabla_+ \omega^{r'} = 0 . \quad (5.16)$$

Next

$$\mathcal{D}_- \omega_{ij}^{r'} = \nabla_- \omega_{ij}^{r'} - H_-^k {}_{[i} \omega_{j]k}^{r'} + 2 \mathcal{C}_-^{s'} \epsilon^{r'}{}_{s't'} \omega_{ij}^{t'} = 0 . \quad (5.17)$$

Since H_{-ij} is self-dual, this implies that it can be written as

$$H_{-ij} = w_{r'} \omega_{ij}^{r'} , \quad (5.18)$$

for some functions $w_{r'}$. Thus

$$\nabla_- \omega_{ij}^{r'} + w^{s'} \epsilon^{r'}{}_{s't'} \omega_{ij}^{t'} + 2 \mathcal{C}_-^{s'} \epsilon^{r'}{}_{s't'} \omega_{ij}^{t'} = 0 . \quad (5.19)$$

This is interpreted as a condition which relates $\mathcal{C}_-^{s'}$ to the H_{-ij} components of the torsion. As a result, it can be solved to express H_{-ij} in terms of other fields and the geometry of spacetime.

To determine the conditions imposed on the geometry from the gravitino KSE in directions transverse to (e^+, e^-) , observe that a generic metric connection in 4 dimensions has holonomy contained in $Sp(1) \cdot Sp(1)$. Thus the only condition required is the identification of $Sp(1)$ part of the $\hat{\nabla}$ spacetime connection with the $Sp(1)$ part of induced connection from the Quaternionic Kähler manifold \mathcal{Q} of the hyper-multiplets. This also follows from the integrability conditions (5.10).

Thus to summarize, the spacetime admits a null Killing vector field X whose rotation in the directions transverse to the light-cone is anti-self-dual, ie

$$de_{ij}^- = -\frac{1}{2} \epsilon_{ij}{}^{kl} de_{kl}^- . \quad (5.20)$$

The geometry is restricted by (5.16). Furthermore, (5.19) relates the self-dual H_{-ij} component of the torsion to the \mathcal{C}_- component of the induced $Sp(1)$ connection from the Quaternionic Kähler manifold of the hyper-multiplets. The remaining conditions are given by the integrability conditions (5.10). The metric and torsion of the spacetime can be written as

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j , \\ H &= e^+ \wedge de^- - \left(\frac{1}{16} \omega_{kl}^{r'} \nabla_- \omega^{s'kl} \epsilon_{r's't'} + \mathcal{C}_-^{t'} \right) \omega_{t'ij} e^- \wedge e^i \wedge e^j \\ &\quad - \frac{1}{3!} (de^-)_{-\ell} \epsilon_{ijk}^\ell e^i \wedge e^j \wedge e^k . \end{aligned} \quad (5.21)$$

These are the full set of conditions on the fields and geometry of spacetime for the gravitino KSE to admit a parallel spinor.

5.1.2 Gaugini

Substituting, the Killing spinor $1 + e_{1234}$ into the gaugini KSE, one finds that the solution to the linear system is

$$F_{+i}^{\mathfrak{m}} = F_{+-}^{\mathfrak{m}} = 0, \quad F_{\alpha}^{\mathfrak{m}} + i\mu_1^{\mathfrak{m}} = 0, \quad 2F_{12}^{\mathfrak{m}} + \mu_2^{\mathfrak{m}} - i\mu_3^{\mathfrak{m}} = 0. \quad (5.22)$$

As a result, we find that the gauge field can be written

$$F^{\mathfrak{m}} = F_{-i}^{\mathfrak{m}} e^{-} \wedge e^i + \frac{1}{2} \mu_r^{\mathfrak{m}} \omega^{r'} + (F^{\text{asd}})^{\mathfrak{m}}, \quad (5.23)$$

where $F_{-i}^{\mathfrak{m}}$ and the anti-self-dual components F_{ij}^{asd} are not restricted by the KSEs. The self-dual part is completely determined in terms of the moment maps μ .

5.1.3 Tensorini

A direct computation of the tensorini KSEs on the spinor $1 + e_{1234}$ reveals that

$$\begin{aligned} T_+^I &= 0, \quad H_{+\alpha}^I{}^{\alpha} = H_{+\alpha\beta}^I = 0, \\ T_{\bar{\alpha}}^I - \frac{1}{2} H_{-+\bar{\alpha}}^I - \frac{1}{2} H_{\bar{\alpha}\beta}^I{}^{\beta} &= 0. \end{aligned} \quad (5.24)$$

Note that the tensorini KSEs commute with the Clifford algebra operations $\rho^{r'}$ in (3.17). As a result, if the tensorini KSE admits a solution ϵ , then $\rho^{r'}\epsilon$ also solve the KSE. As a result, the four spinors

$$1 + e_{1234}, \quad \rho^{r'}(1 + e_{1234}), \quad r' = 1, 2, 3, \quad (5.25)$$

are solutions to the tensorini KSE.

The 3-form field strengths are self-dual in 6 dimensions. This implies that

$$H_{-\alpha\beta}^I = H_{-\alpha}^I{}^{\alpha} = 0, \quad H_{-+\bar{\alpha}}^I - H_{\bar{\alpha}\beta}^I{}^{\beta} = 0, \quad H_{+1\bar{1}}^I - H_{+2\bar{2}}^I = 0, \quad H_{+1\bar{2}}^I = 0. \quad (5.26)$$

Combining these conditions with those from the tensorini KSE, one finds that

$$H_{+ij}^I = 0. \quad (5.27)$$

Moreover (5.26) implies that H_{-ij}^I is *anti-self-dual* in the directions transverse to (e^+, e^-) and this component is not otherwise restricted by the KSEs. Therefore, the solution of the KSEs can be expressed as

$$\begin{aligned} T^I &= T_-^I e^{-} + T_i^I e^i, \\ H^I &= \frac{1}{2} H_{-ij}^I e^{-} \wedge e^i \wedge e^j + T_i^I e^{-} \wedge e^+ \wedge e^i - \frac{1}{3!} T_{\ell}^I \epsilon^{\ell}_{ijk} e^i \wedge e^j \wedge e^k. \end{aligned} \quad (5.28)$$

In addition, $T_i^I = x_{\underline{t}}^I \partial_i v^{\underline{t}}$. Substituting this in (5.28) all components of H^I apart from H_{-ij}^I are expressed in terms of the tensor multiplet scalars.

5.1.4 Hyperini

To solve the hyperini KSE, one has to identify the ϵ_i components of the Killing spinor in the context of spinorial geometry. In our notation $\epsilon^1 = 1$ and $\epsilon^2 = e_{1234}$ and since $\epsilon_1 = -\epsilon^2$ and $\epsilon_2 = \Gamma_{34}\epsilon^1$ as in (3.19), one has $\epsilon_1 = -e_{1234}$ and $\epsilon_2 = e_{34}$. Substituting these into the KSE, one finds the conditions

$$V_+^{a\mathbf{i}} = 0, \quad -V_1^{a\mathbf{1}} + V_2^{a\mathbf{2}} = 0, \quad V_2^{a\mathbf{1}} + V_1^{a\mathbf{2}} = 0, \quad (5.29)$$

where we have set $\mathbf{i} = \underline{1}, \underline{2}$ to distinguish the range of the \mathbf{i} index from the range of the holomorphic index $\alpha = 1, 2$ of the spacetime. The conditions (5.29) can be expressed in terms of the hyper-multiplet scalars as

$$D_+ q^M = 0, \quad (\tau^i)^{\mathbf{i}}{}_{\mathbf{j}} D_i q^M E^{\mathbf{j}a}{}_M = 0, \quad (5.30)$$

where $(\tau^i) = (-i\sigma_{r'}, 1_{2 \times 2})$ and $\sigma_{r'}$ are the Pauli matrices. In the gauge $A_+ = 0$, the fields q^M do not dependent on the coordinate u adapted to the Killing vector field $X = \partial_u$ as expected. The last condition in (5.30) can equivalently be written in a coordinate basis as

$$(\mathfrak{J}^i)^M{}_N D_i q^N = 0, \quad (5.31)$$

where $(\mathfrak{J}^i) = (I_1, I_2, I_3, 1_{4n \times 4n})$.

5.2 N=2 non-compact

There are two cases with $N = 2$ supersymmetry distinguished by the isotropy group of the Killing spinors. If the isotropy group is non-compact $Sp(1) \cdot U(1) \ltimes \mathbb{H}$, the two Killing spinors are

$$\epsilon_1 = 1 + e_{1234}, \quad \epsilon_2 = i(1 - e_{1234}) = \rho^1 \epsilon_1. \quad (5.32)$$

The additional conditions on the fields which arise from the second Killing spinor can be expressed as the requirement that the KSEs must commute with the Clifford algebra operation ρ^1 .

5.2.1 Gravitino

It is clear that the gravitino KSE commutes with ρ^1 , iff

$$\mathcal{C}_\mu^2 = \mathcal{C}_\mu^3 = 0. \quad (5.33)$$

The form spinor bi-linears are given in (5.9) and so the full content of gravitino KSE can be expressed as

$$\begin{aligned} \hat{\nabla} e^- &= 0, \quad \hat{\nabla}(e^- \wedge \omega) = 0, \quad \hat{\nabla}(e^- \wedge \omega^2) - 2\mathcal{C}e^- \wedge \omega^3 = 0, \\ \hat{\nabla}(e^- \wedge \omega^3) + 2\mathcal{C}e^- \wedge \omega^2 &= 0, \end{aligned} \quad (5.34)$$

where $\omega = \omega^1$ and $\mathcal{C} = \mathcal{C}^1$ and $\omega^{r'}$ are given in (5.8).

These conditions can be solved as follows. The first implies the conditions (5.12), ie that the 1-form e^- is associated with a null Killing vector field. The remaining conditions can be solved yielding the geometric conditions

$$\begin{aligned}\nabla_+ \omega_{ij}^{r'} &= 0, \quad (de^-)_{-\ell} \epsilon_{ijk}^\ell = (i_J \tilde{d}\omega)_{ijk}, \\ \nabla_- \omega_{ij}^2 - \nabla_- \omega_{k[i}^1 (J_3)^k_{j]} - \frac{1}{4} \nabla_- \omega_{k\ell}^2 \omega^{3k\ell} \omega_{ij}^3 &= 0, \\ \nabla_- \omega_{ij}^3 + \nabla_- \omega_{k[i}^1 (J_2)^k_{j]} + \frac{1}{4} \nabla_- \omega_{k\ell}^2 \omega^{3k\ell} \omega_{ij}^2 &= 0,\end{aligned}\tag{5.35}$$

and the integrability conditions

$$\begin{aligned}\hat{R}_{\mu_1\mu_2,+\nu} &= 0, \quad \hat{R}_{\mu_1\mu_2,ki} J^k_j - \hat{R}_{\mu\nu,kj} J^k_i = 0, \\ -\hat{R}_{\mu_1\mu_2,ki} (J_2)^k_j + \hat{R}_{\mu_1\mu_2,kj} (J_2)^k_i - 2\mathcal{F}_{\mu_1\mu_2} \omega_{ij}^3 &= 0,\end{aligned}\tag{5.36}$$

where we have set $J = J_1$ as this is distinguished from J_2 and J_3 .

To derive the first condition in (5.35) we have used $D_+ q^M = 0$ which follows from the hyperini KSE as explained in the $N = 1$ case. The second condition in (5.35) arises from the solution of the second condition in (5.5). The integrability conditions (5.36) first restrict the holonomy of the $\hat{\nabla}$ connection along the directions transverse to (e^+, e^-) to lie in $U(2) = Sp(1) \cdot U(1)$ and the last condition identifies the $U(1)$ part of the curvature \hat{R} with the curvature of \mathcal{C} .

Moreover, one finds the following expressions for some components of the fields

$$H_{-ij} = -\nabla_- \omega_{ik} I^k_j, \quad \mathcal{C}_- = \frac{1}{8} \nabla_- \omega_{ij}^2 \omega^{3ij}.\tag{5.37}$$

To summarize, the gravitino KSE implies that the metric and H can be written as

$$\begin{aligned}ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j, \\ H &= e^+ \wedge de^- - \nabla_- \omega_{ik} I^k_j e^- \wedge e^i \wedge e^j - \frac{1}{3!} (de^-)_{-\ell} \epsilon_{ijk}^\ell e^i \wedge e^j \wedge e^k.\end{aligned}\tag{5.38}$$

This concludes the description of the conditions that arise from the gravitino KSE.

5.2.2 Gaugini

The gaugini KSE commutes with ρ^1 , iff

$$\mu_2 = \mu_3 = 0.\tag{5.39}$$

As a result, we have that

$$F^{\mathfrak{m}} = F_{-i}^{\mathfrak{m}} e^- \wedge e^i + \frac{1}{2} \mu^{\mathfrak{m}} \omega + (F^{\text{asd}})^{\mathfrak{m}}, \quad \mu^2 = \mu^3 = 0,\tag{5.40}$$

where $\mu = \mu^1$.

5.2.3 Tensorini

A direct substitution of the second Killing spinor into the tensorini KSEs reveals that there are no additional conditions to those given in (5.24). As we have mentioned the tensorini KSEs commute with all ρ Clifford algebra operations.

5.2.4 Hyperini

Combining the restrictions imposed by the second Killing spinor with those presented in (5.29) for the first Killing spinor, one finds

$$V_+^{a\mathbf{i}} = 0 , \quad V_\alpha^{a\mathbf{1}} = 0 , \quad V_{\bar{\alpha}}^{a\mathbf{2}} = 0 , \quad (5.41)$$

where again $\mathbf{i} = \underline{1}, \underline{2}$. These equations can be rewritten as

$$D_+ q^M = 0 , \quad (I_3)^M{}_N D_i q^N = J^j{}_i D_j q^M . \quad (5.42)$$

The last equation is a Cauchy-Riemann type of equation, ie in the absence of gauge fields, q 's satisfy a holomorphicity condition with respect to the pair of complex structures (J, I_3) .

5.3 N=2 compact

5.3.1 Gravitino

The Killing spinors are $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = e_{15} + e_{2345}$ as stated in table 1. It is straightforward to find that a basis in the form spinor bi-linears is given by the 1-forms

$$\lambda^a , \quad a = -, +, 1 ; \quad e^i , \quad i = 1, 2, 3 , \quad (5.43)$$

where we have appropriately relabeled the range of the indices a and i . Note that the original labeling which arises from the identification of gamma matrices γ in (3.14) is $a = -, +, 1$ and $i = 2, 6, 7$.

The conditions implied by the gravitino KSE can be rewritten as

$$\begin{aligned} \hat{\nabla}_\mu \lambda^a &= 0 , \\ \hat{\nabla}_\mu e^i + 2\epsilon^i{}_{jk} \mathcal{C}_\mu^j e^k &= 0 , \end{aligned} \quad (5.44)$$

where an appropriate identification is chosen between the indices r', s' and t' which appear in (5.5) and i, j and k followed by an appropriate identification of components of \mathcal{C} .

The three 1-forms λ^a are parallel with respect to a connection with skew symmetric torsion on the spacetime. As a result, they are no-where vanishing and their inner product $\eta^{ab} = g(\lambda^a, \lambda^b)$ is constant. In fact, (λ^a, e^i) can be used as a frame on the spacetime and write the metric as

$$ds^2 = \eta_{ab} \lambda^a \lambda^b + \delta_{ij} e^i e^j . \quad (5.45)$$

It is clear that the spacetime admits a $3 + 3$ “split”. In particular, the tangent space, TM , of spacetime decomposes as

$$TM = I + \xi , \quad (5.46)$$

where I is a topologically trivial vector bundle spanned by the vector fields X_a associated to the three 1-forms λ^a .

To continue, let us focus on the first equation in (5.44). This implies that

$$\mathcal{L}_{X_a} g = 0 , \quad d\lambda^a = \eta^{ab} i_b H \quad (5.47)$$

ie X_a are Killing and the $i_b H$ component of H is expressed as the exterior derivative of λ^a .

We shall not deal with the most general case here. This has been done in [20]. Instead, we shall assume that the algebra of three Killing vector fields X_a closes. This together with the anti-self duality of H implies that the only non-vanishing components of H are H_{abc} and H_{ijk} , and

$$d\lambda^a = \frac{1}{2} H^a{}_{bc} \lambda^b \wedge \lambda^c, \quad H_{abc} \epsilon^{abc} = H_{ijk} \epsilon^{ijk} \quad (5.48)$$

for some choice of orientation in I and ξ such that $\epsilon_{abcijk} = \epsilon_{abc} \epsilon_{ijk}$. The first condition implies that the spacetime metrically splits locally into a product $G \times \Sigma^3$, where G is a Lorentzian 3-dimensional group and Σ^3 is a 3-dimensional manifold. In fact, the Lie algebra of G is

$$\mathbb{R}^{2,1}, \quad \mathfrak{sl}(2, \mathbb{R}), \quad (5.49)$$

where we have used the classification of Lorentzian Lie algebras in [46, 47].

It remains to investigate the geometry of Σ^3 . Σ^3 is induced with a metric and a 3-form field strength as

$$d\tilde{s}^2(\Sigma^3) = \delta_{ij} e^i e^j, \quad \tilde{H} = \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k, \quad (5.50)$$

which in turn define a connection with skew-symmetric torsion $\hat{\tilde{\nabla}}$. Taking the integrability of the second condition in (5.44), we find that the curvature of $\hat{\tilde{\nabla}}$ is

$$\hat{\tilde{R}}_{i_1 i_2, j_1 j_2} = -2 \mathcal{F}_{i_1 i_2}^k \epsilon_{k j_1 j_2}, \quad (5.51)$$

where we have used $\mathcal{C}_a = 0$ which follows from the hyperini KSE later. This condition implies that the curvature of $\hat{\tilde{\nabla}}$ is given in terms of $Sp(1)$ part of the curvature of the Quaternionic Kähler manifold \mathcal{Q} of the hyper-multiplet scalars induced on the spacetime.

To summarize, the spacetime is locally a product $G \times \Sigma^3$, where G is a 3-dimensional Lorentzian group and Σ^3 is a Riemannian manifold, such that one has

$$ds^2 = \eta_{ab} \lambda^a \lambda^b + \delta_{ij} e^i e^j, \quad H = \frac{1}{3!} H_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c + \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k, \quad (5.52)$$

provided that the conditions (5.48) and (5.51) hold.

5.3.2 Gaugini

Evaluating the gaugini KSE on $e_{15} + e_{2345}$ and combining the resulting conditions with those of (5.22) that are derived from evaluating the gaugini KSE on the first spinor $1 + e_{1234}$, we get that

$$F^m = -\frac{1}{2} \epsilon_{ijk} \mu^{mk} e^i \wedge e^j, \quad (5.53)$$

where again we have appropriately identify the $r', s', t' = 1, 2, 3$ indices of the moment maps with $i, j, k = 1, 2, 3$, ie with those of the frame on Σ^3 . Therefore, the curvature field strengths have support on Σ^3 and are completely determined in terms of the moment maps μ .

5.3.3 Tensorini

Substituting $e_{15} + e_{2345}$ into the tensorini KSEs and comparing the resulting conditions with those derived in (5.24) which arise from evaluating the same KSEs on $1 + e_{1234}$, and using the self-duality of H^I (5.26), one finds that

$$T_\mu^I = 0 \ , \quad H_{\mu\nu\rho}^I = 0 \ . \quad (5.54)$$

Expressing of T and H^I in terms of the physical fields (3.6), one finds that the scalars are constant and 3-form field strengths of the tensor multiplet vanish.

5.4 Hyperini

Evaluating the hyperini KSE on $e_{15} + e_{2345}$ and comparing the results with those of (5.29) which arise from evaluating the same KSE on the first spinor $1 + e_{1234}$, we find that

$$D_a q^M = 0 \ , \quad D_i q^M = -\epsilon_i^{jk} (I_j)^M{}_N D_k q^N \ . \quad (5.55)$$

In the gauge that $A_a = 0$ which can always be chosen locally as $F_{ab}^m = 0$ from the gaugini KSE, one concludes that q does not depend on the coordinates of the group G .

6 N=4 non-compact

The four Killing spinors with isotropy group $Sp(1) \ltimes \mathbb{H}$ of table 1 can be rewritten as

$$1 + e_{1234} \ , \quad \rho^1(1 + e_{1234}) \ , \quad \rho^2(1 + e_{1234}) \ , \quad \rho^3(1 + e_{1234}) \ . \quad (6.1)$$

Therefore for the KSEs to admit these as Killing spinors they must commute with the Clifford algebra operations $\rho^{r'}$. This together with the conditions we have found for backgrounds to preserve one supersymmetry give the full set of conditions on the fields in this case.

6.0.1 Gravitino

The gravitino KSE commutes with the $\rho^{r'}$ operations iff $\mathcal{C} = 0$. The spinor bilinears are in (5.7) but now their conditions read

$$\hat{\nabla} e^- = 0 \ , \quad \hat{\nabla}(e^- \wedge \omega^{r'}) = 0 \ . \quad (6.2)$$

Following similar steps to those of the non-compact $N = 1$ and $N = 2$ cases, the fields can be expressed as

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j \ , \\ H &= e^+ \wedge de^- - \frac{1}{16} \omega_{kl}^{r'} \nabla_- \omega^{s'kl} \epsilon_{r's'}{}^{t'} \omega_{t'ij} e^- \wedge e^i \wedge e^j \\ &\quad - \frac{1}{3!} (de^-)_{-\ell} \epsilon^\ell{}_{ijk} e^i \wedge e^j \wedge e^k \ . \end{aligned} \quad (6.3)$$

It remains to present the geometric conditions on the spacetime. These are

$$\begin{aligned} \nabla_+ \omega^{r'} &= 0, \quad de_{ij}^- = -\frac{1}{2} \epsilon_{ij}^{kl} de_{kl}^-, \\ de_{-j}^- \epsilon^j_{i_1 i_2 i_3} &= (i_{J_{r'}} \tilde{d}\omega^{r'})_{i_1 i_2 i_3}, \quad (\text{no } r' \text{ summation}). \end{aligned} \quad (6.4)$$

To derive these, we have solved (6.2) and applied the anti-self-duality of H .

6.0.2 Gaugini

The KSEs commute with $\rho^{r'}$, iff

$$\mu_1 = \mu_2 = \mu_3 = 0. \quad (6.5)$$

These are in addition to the conditions given in (5.22). Thus, we have that

$$F^{\mathfrak{m}} = F_{-i}^{\mathfrak{m}} e^- \wedge e^i + (F^{\text{asd}})^{\mathfrak{m}}. \quad (6.6)$$

6.0.3 Tensorini

The tensorini KSE commutes with the Clifford algebra operations $\rho^{r'}$. Thus there are no additional conditions to those given in (5.24)

6.0.4 Hyperini

The conditions which arise from the hyperini KSEs are

$$D_+ q^M = D_i q^M = 0 \quad (6.7)$$

Therefore the only non vanishing component of the derivative on the scalars is $D_- q^M$.

6.0.5 N=3 descendant

Unlike all other cases, the $N = 4$ backgrounds with $Sp(1) \ltimes \mathbb{H}$ -invariant parallel spinors exhibit an independent descendant with 3 supersymmetries. The conditions for this can be easily found by evaluating the hyperini KSEs on the three spinors (4.7). The conditions on the scalar q are

$$D_+ q^M = 0, \quad (J_{r'})^i_j D_i q^M = (I_{r'})^M_N D_j q^N, \quad (6.8)$$

where J_r are the complex structures transverse to (e^+, e^-) associated with 2-form bilinears $\omega^{r'}$ and $I_{r'}$ is the quaternionic structure on the scalar manifold \mathcal{Q} of the hyper-multiplets. The above condition in the absence of gauge fields implies that q 's are locally quaternionic maps.

6.1 N=4 compact

The Killing spinors are the $U(1)$ -invariant spinors of table 1. These can be rewritten as

$$1 + e_{1234}, \quad e_{15} + e_{2345}, \quad \rho^1(1 + e_{1234}), \quad \rho^1(e_{15} + e_{2345}). \quad (6.9)$$

Thus the conditions on the fields that arise from the KSEs are those we have found for the $Sp(1)$ -invariant Killing spinors, and those required for the KSEs to commute with the Clifford algebra operation ρ^1 .

6.1.1 Gravitino

The Clifford algebra operation ρ^1 commutes with the gravitino KSE provided that $\mathcal{C}^2 = \mathcal{C}^3 = 0$. A basis for algebraically independent spinor bilinears is spanned by the 1-forms

$$\lambda^a, \quad a = -, +, 1, \bar{1}, \quad e^i, \quad i = 2, \bar{2}. \quad (6.10)$$

The conditions that arise from gravitino KSE can be rewritten as

$$\hat{\nabla} \lambda^a = 0, \quad \hat{\nabla} e^i - 2\mathcal{C} \epsilon^i_j e^j = 0, \quad (6.11)$$

where we have set $\mathcal{C} = \mathcal{C}^1$.

The first condition in (6.11) implies that

$$\mathcal{L}_{X_a} g = 0, \quad i_a H = \eta_{ab} d\lambda^b, \quad (6.12)$$

ie the vector fields X_a associated to λ^a are Killing and that the $i_a H$ component of H is given in terms of the exterior derivative of λ^a , where $\eta_{ab} = g(X_a, X_b)$ is constant. It is clear that the spacetime admits a $4+2$ split. In particular, the tangent space $TM = I \oplus \xi$, where now I is a rank 4 trivial vector bundle spanned by the 4 Killing vectors X_a .

To continue, we assume that the algebra of the four Killing vector field closes, ie $H_{abi} = 0$. The more general case without this assumption has been presented in [20]. The Lorentzian 4-dimensional Lie algebras have been classified and so the algebra of Killing vector fields X_a must be isomorphic [46, 47] to one of the following

$$\mathbb{R}^{3,1}, \quad \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1), \quad \mathbb{R} \oplus \mathfrak{su}(2), \quad \mathfrak{cw}_4. \quad (6.13)$$

Furthermore the anti-self duality of H implies that

$$H_{aij} = \frac{1}{3!} \epsilon_{ij} \epsilon_a^{b_1 b_2 b_3} H_{b_1 b_2 b_3}, \quad (6.14)$$

where $\epsilon_{abcdij} = \epsilon_{abcd} \epsilon_{ij}$.

Next, we have that

$$d\lambda^a - \frac{1}{2} H^a_{bc} \lambda^b \wedge \lambda^c = \frac{1}{2} H^a_{ij} e^i \wedge e^j, \quad (6.15)$$

where H_{abc} are the structure constants of the Lie algebra of the four Killing vector fields. Locally the spacetime can be thought of as a principal bundle with fibre group that has a Lie algebra as in (6.13), base space a 2-dimensional manifold Σ^2 and principal bundle connection λ^a . In such a case, the rhs of (6.15) is the curvature of λ which measures the twist of the fibre over the base space. Since the curvature does not vanish the splitting of spacetime is not a product. This is unlike the $3+3$ splitting of the $N=2$ backgrounds which is a product. The last condition in (6.11) identifies the spacetime connection along the directions transverse to the Killing vectors with a $U(1)$ component of the induced $Sp(1)$ quaternionic Kähler connection. This can also be seen by investigating the integrability conditions of (6.11). In particular, one finds that the only non-vanishing components of the \hat{R} curvature of spacetime are

$$\hat{R}_{i_1 i_2, j_1 j_2} = -2 \mathcal{F}_{i_1 i_2} \epsilon_{j_1 j_2}. \quad (6.16)$$

where we have anticipated the results from the hyperini KSE that $\mathcal{C}_a = 0$.

To summarize, the metric and 3-form field strengths are

$$\begin{aligned} ds^2 &= \eta_{ab} \lambda^a \lambda^b + \delta_{ij} e^i e^j , \\ H &= \frac{1}{3!} H_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c + \frac{1}{2 \cdot 3!} \epsilon_{ij} \epsilon_a{}^{b_1 b_2 b_3} H_{b_1 b_2 b_3} e^a \wedge e^i \wedge e^j , \end{aligned} \quad (6.17)$$

and the geometric conditions are given in (6.12) and (6.16).

6.1.2 Gaugini

The gaugini KSE commutes with ρ^1 iff $\mu^2 = \mu^3 = 0$. Combining this with (5.53), one finds

$$F^m = \frac{1}{2} \mu^m \epsilon_{ij} e^i \wedge e^j , \quad (6.18)$$

where $\mu = \mu^1$.

6.1.3 Tensorini

The tensorini KSE commutes with all the Clifford algebra $\rho^{r'}$ operators. Since both $1 + e_{1234}$ and $e_{15} + e_{2345}$ are Killing spinors, one concludes that all 8 supersymmetries are preserved. Thus $T^I = H^I = 0$ as in (5.54). In turn, the tensorini multiplet scalars are constant and the 3-form field strengths vanish.

6.1.4 Hyperini

Evaluating the hyperini KSEs on the Killing spinors, one finds

$$D_a q^M = 0 , \quad a = -, +, 1, \bar{1} , \quad i D_2 q^M = (I_3)^M{}_N D_2 q^N . \quad (6.19)$$

Clearly, the scalar fields q do not depend on 4 spacetime directions in the gauge $A_a = 0$. The last condition is Cauchy-Riemann type of equations along the remaining two directions.

6.2 Trivial isotropy group

Backgrounds with parallel spinors which have a trivial isotropy group admit 8 parallel spinors. The spacetime is a Lorentzian Lie group with anti-self-dual structure constants. These have been classified in a similar context in [41]. In particular, the spacetime is locally isometric to

$$\mathbb{R}^{5,1} , \quad AdS_3 \times S^3 , \quad CW_6 , \quad (6.20)$$

where the radii of AdS_3 and S^3 are equal, and the structure constants of CW_6 are given by a constant self-dual 2-form on \mathbb{R}^4 . Moreover

$$\mathcal{F}(\mathcal{C}) = 0 . \quad (6.21)$$

This concludes the conditions which arise from the gravitino KSE.

The gaugini KSEs imply that the gauge field strengths vanish and that $\mu^{r'} = 0$. The tensorini KSEs imply that the 3-form field strengths vanish and the tensor multiplet scalars are constants. Similar hyperini KSEs imply that the scalars q are constant. In turn using (3.6), the latter gives $\mathcal{C} = 0$.

7 Black hole horizons

It is well known that the black hole uniqueness theorems in four dimensions [48]-[54] do not extend to five and higher. Specifically in five dimensions, apart from spherical supersymmetric black holes [55], there also exist black holes with near horizon topology $S^1 \times S^2$, the black rings [56, 57]. In more than five dimensions, it is expected that there are black holes with exotic horizon topologies [58]-[62].

The progress that has made towards understanding the geometry of all solutions to the KSEs of supergravity theories raises the possibility that all supersymmetric black hole solutions can be classified. So far this goal has not been attained but some significant progress has been made towards the classification of all near horizon black hole geometries, see [63] for a recent review and [64] for brane horizons. Results in this direction include the identification of all near horizon geometries of simple 5- and 6-dimensional supergravities [56, 65]. In addition, all near horizon geometries of 4-dimensional $\mathcal{N} = 1$ supergravity coupled to any number of vector and scalar multiplets have been classified [66] and a similar result has been established for heterotic horizons [67]. The geometries of IIB and 11-dimensional supergravity horizons have been investigated in [68, 69]. More recently, it has been conjectured that supersymmetric near horizon black hole geometries exhibit supersymmetry enhancement and are invariant under an $SL(2, \mathbb{R})$ symmetry. The latter property is significant as it illustrates the close relationship between near horizon geometries and conformal symmetry. The conjecture has been proven for a number of theories in [70] and has been used to show that there are no asymptotically AdS_5 supersymmetric black rings [70, 71]. The latter generalizes the result of [72] proven under stronger symmetry assumptions.

One of the applications of the solution of the KSEs of (1,0) supergravity theory coupled to any number of vector, tensor, and scalar multiplets is in the context of the near horizon geometries of 6-dimensional black holes which preserve at least one supersymmetry. In particular, one can show that 6-dimensional (1,0) supergravity coupled to any number of tensor and scalar multiplets has two classes of near horizon geometries. One is locally isometric to $AdS_3 \times \Sigma^3$, where Σ^3 is diffeomorphic to S^3 , and the other is locally isometric to $\mathbb{R}^{1,1} \times \mathcal{S}$, where the geometry of \mathcal{S} depends on the hypermultiplet scalars. These results have been established in [32] and in what follows we shall describe some of the key steps in the proof.

In this review, the main focus is on the $AdS_3 \times \Sigma^3$ class. This is because it exhibits some attractive properties like supersymmetry enhancement and a $\times^2 SL(2, \mathbb{R})$ invariance which, as it has been mentioned, are now conjectured to be properties of supersymmetric horizons. These horizons preserve 2, 4 and 8 supersymmetries. In the latter case, they are locally isometric to $AdS_3 \times S^3$ with the radii of the two subspaces equal.

7.1 Supersymmetric horizons

7.1.1 Near horizon geometry

For the application to near horizon geometry of extreme black holes, we shall consider (1,0) supergravity theories coupled to any number of tensor and scalar multiplets. The fields can be written in Gaussian null coordinates [73]. Such coordinates always exists for extreme, smooth, Killing horizons. In these coordinates, the near horizon fields can be expressed as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j, \\ G^r &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge (d_h S^r - N^r) - r\mathbf{e}^+ \wedge (d_h N^r + S^r dh) + dW^r, \\ q^I &= q^I(y), \quad \phi = \phi(y), \end{aligned} \quad (7.1)$$

where

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + rh + r^2 \Delta du, \quad \mathbf{e}^i = e^i_P dy^P, \quad (7.2)$$

and $d_h S^r = dS^r - hS^r$ and $d_h N^r = dN^r - h \wedge N^r$. The spacetime has coordinates (r, u, y^P) . The black hole horizon section \mathcal{S} is the co-dimension 2 subspace $r = u = 0$ and it is assumed to be *compact, connected, and without boundary*. The dependence of fields on light-cone coordinates (r, u) is explicitly given. In addition, dW^r are 3-forms, h, N^r are 1-forms, and S^r are scalars on the horizon section \mathcal{S} and depend only on the coordinates y . \mathbf{e}^i is a frame on \mathcal{S} and depends only on y as well. Both the tensor and hyper-multiplet scalars depend only on the coordinates of \mathcal{S} .

To find the supersymmetric horizons of 6-dimensional (1,0) supergravity, one has to solve both the field and KSEs of the theory for the fields given in (7.1). We shall proceed with the solution of KSEs.

7.1.2 Solution of KSEs

To continue, we substitute (7.1) into the KSEs (3.12) and assume that the backgrounds preserve at least one supersymmetry. Furthermore, we identify the stationary Killing vector field ∂_u of the near horizon geometry with the Killing vector constructed as a Killing spinor bilinear. This may appear as an additional restriction but this is not the case as it has been established for the analogous case of heterotic horizons in [70]. Since the vector Killing spinor bilinear is null, one concludes that $\Delta = 0$. Moreover, it turns out that the Killing spinor can always be chosen [32] as

$$\epsilon = 1 + e_{1234}. \quad (7.3)$$

In such a case, a direct comparison with the expression for the fields for $N = 1$ backgrounds in (5.21), (5.28) and (5.30) implies that the fields can be rewritten as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j, \\ H &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge h + r\mathbf{e}^+ \wedge dh - \frac{1}{3!}h_\ell \epsilon^\ell_{ijk} \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k, \\ H^I &= T^I_i \mathbf{e}^- \wedge \mathbf{e}^+ \wedge \mathbf{e}^i - \frac{1}{3!}T^I_\ell \epsilon^\ell_{ijk} \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k. \end{aligned}$$

$$q^M = q^M(y) , \quad \phi = \phi(y) , \quad (7.4)$$

where we have used the duality relations of the 3-form field strengths. In addition the anti-self duality of H requires that

$$dh_{ij} = -\frac{1}{2}\epsilon_{ij}{}^{kl}dh_{kl} . \quad (7.5)$$

It is clear that H is entirely determined in terms of h while H^I is entirely determined in terms of the scalars ϕ of the tensor multiplets.

After, rewriting of the fields as in (7.4) and establishing that the Killing spinor is (7.3), the gravitino KSE gives

$$\tilde{\mathcal{D}}_i(1 + e_{1234}) = 0 , \quad (7.6)$$

where

$$\tilde{\mathcal{D}}_i = \hat{\nabla}_i + \mathcal{C}_i{}^{r'} \rho_{r'} , \quad (7.7)$$

and $\hat{\nabla}$ is the connection on \mathcal{S} with skew-symmetric torsion $-\star_4 h$. This is just the restriction of the gravitino KSE on \mathcal{S} . One can unveil the geometric content of this equation by considering the twisted Hermitian 2-forms $\omega^1, \omega^2, \omega^3$ in (5.8) constructed as Killing spinor bi-linears which are now restricted on \mathcal{S} . Then, the integrability condition of (7.6) can be expressed as

$$-\hat{R}_{mn,}{}^k{}_i \omega^{r'}{}_{kj} + (j, i) + 2\mathcal{F}_{mn}^{s'} \epsilon^{r'}{}_{s't'} \omega_{ij}^{t'} = 0 , \quad (7.8)$$

where

$$\mathcal{F}_{mn}^{s'} = \partial_m q^M \partial_n q^N \mathcal{F}_{MN}^{s'} . \quad (7.9)$$

This integrability condition identifies the $Sp(1) \subset Sp(1) \cdot Sp(1)$ component of the curvature \hat{R} of the 4-dimensional manifold \mathcal{S} with the pull back with respect to q of the $Sp(1)$ component of the curvature of the Quaternionic Kähler manifold \mathcal{Q} . The restriction imposed on the geometry of \mathcal{S} by (7.8) depends on the scalars q^M . In particular, if q^M are constant, then $\mathcal{F}_{mn} = 0$ and (7.8) implies that \mathcal{S} is an HKT manifold [74].

There are no additional conditions arising from the tensorini KSE. The hyperini KSE requires that q satisfy (5.30). We shall return to the above conditions imposed by the KSEs after imposing the restrictions on the fields implied by the field equations of the theory and the compactness of \mathcal{S} .

7.2 Horizons with $h \neq 0$ and holonomy reduction

7.2.1 An application of maximum principle

There are two classes of horizons to consider depending on whether or not h vanishes. First, we shall consider only the class that $h \neq 0$. If $h \neq 0$, we demonstrate that the

number of supersymmetries preserved by the near horizon geometries is always even. For this we shall use the results we have obtained from the KSEs for horizons preserving one supersymmetry and the field equations of the theory. The methodology we shall follow to prove this is to compute $\tilde{\nabla}^2 h^2$ and apply the maximum principle utilizing the compactness of \mathcal{S} . In particular, one can establish [32] that

$$\tilde{\nabla}^2 h^2 + h^i \tilde{\nabla}_i h^2 = 2\tilde{\nabla}^i h^j \tilde{\nabla}_i h_j + 4\partial_i q^M \partial_j q^N g_{MN} h^i h^j , \quad (7.10)$$

where $\tilde{\nabla}$ is the Levi-Civita connection of \mathcal{S} with respect to $ds^2(\mathcal{S}) = \delta_{ij} \mathbf{e}^i \mathbf{e}^j$ and \tilde{R} is the associated Ricci tensor. Applying now the maximum principle using the compactness of \mathcal{S} , we find that h^2 is constant and

$$\tilde{\nabla}_i h_j = 0 , \quad h^i \partial_i q^M = 0 . \quad (7.11)$$

To establish the latter equation, we have used that the metric g_{MN} of the Quaternionic Kähler manifold \mathcal{Q} is positive definite. Thus h is a parallel 1-form on \mathcal{S} with respect to the Levi-Civita connection and the scalars of the hyper-multiplets are invariant under the action of h .

The existence of a parallel 1-form on the horizon section \mathcal{S} with respect to the Levi-Civita connection is a strong restriction. First it implies that the holonomy of $\tilde{\nabla}$ is contained in $SO(3) \subset SO(4)$,

$$\text{hol}(\tilde{\nabla}) \subseteq SO(3) . \quad (7.12)$$

Moreover \mathcal{S} metrically (locally) splits into a product $S^1 \times \Sigma^3$, where Σ^3 is a 3-dimensional manifold. In turn, as we shall see, the near horizon geometry is locally a product $AdS_3 \times \Sigma^3$. More elegantly the near horizon geometry admits a supersymmetry enhancement from one supersymmetry to two which we explain later.

To prove (7.10), we first state the field equations of 6-dimensional supergravity in the absence of vector multiplets as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} \varsigma_{rs} G_\mu^r{}^{\lambda\rho} G_\nu^\lambda{}_{\rho} G_{\lambda\rho}^s + \partial_\mu v^x \partial_\nu v_x - 2g_{MN} \partial_\mu q^M \partial_\nu q^N &= 0 , \\ \nabla_\lambda (\varsigma_{rs} G^{s\lambda\mu\nu}) &= 0 , \\ \nabla^\mu \partial_\mu v^x + \frac{1}{6} v_x G^{s\mu\nu\rho} G_{\mu\nu\rho}^r &= 0 , \\ D_\mu \partial^\mu q^M &= 0 , \end{aligned} \quad (7.13)$$

where in the last equation it is understood that the Levi-Civita connections of both the spacetime and the Quaternionic Kähler manifold \mathcal{Q} have been used to covariantize the expression.

Then one finds that

$$\tilde{\nabla}^2 h^2 = 2\tilde{\nabla}^i h^j \tilde{\nabla}_i h_j + 2\tilde{\nabla}^i (dh)_{ij} h^j + 2\tilde{R}_{ij} h^i h^j + 2h^j \tilde{\nabla}_j \tilde{\nabla}_i h^i . \quad (7.14)$$

The proof of this is given in [67]. To proceed, we shall utilize the field equations to rearrange the above expression in such a way that we can apply the maximum principle. Using the Einstein equation and

$$\tilde{R}_{ij} = R_{ij} - \tilde{\nabla}_{(i} h_{j)} + \frac{1}{2} h_i h_j , \quad (7.15)$$

one finds that

$$2\tilde{R}_{ij}h^ih^j = -h^2\partial_k v_{\underline{r}}\partial^k v^{\underline{r}} + 4\partial_i q^M\partial_j q^N g_{MN}h^ih^j - h^i\tilde{\nabla}_i h^2 . \quad (7.16)$$

The $\mu\nu = +-$ component of the field equation $\nabla_\lambda(\zeta_{rs}G^{\lambda\mu\nu})$ together with $H^{i+-} = -h^i$ and $H^{\underline{M}i+-} = T^{\underline{M}}$ give

$$\partial_i v_{\underline{r}}h^i + v_{\underline{r}}\tilde{\nabla}_i h^i + \tilde{\nabla}_i\partial^i v_{\underline{r}} = 0 . \quad (7.17)$$

Acting on the above expression with $v^{\underline{r}}$, we find

$$\tilde{\nabla}_i h^i + v^{\underline{r}}\tilde{\nabla}_i\partial^i v_{\underline{r}} = 0 , \quad (7.18)$$

where we have used $v_{\underline{r}}v^{\underline{r}} = 1$.

The field equation of the scalars of the tensor multiplet gives

$$v_{\underline{r}}\tilde{\nabla}_i\partial^i v^{\underline{r}} = 0 , \quad (7.19)$$

which when combined with (7.18) implies that

$$\tilde{\nabla}_i h^i = 0 . \quad (7.20)$$

In addition (7.19) and $v_{\underline{r}}v^{\underline{r}} = 1$ give

$$\partial_k v_{\underline{r}}\partial^k v^{\underline{r}} = 0 . \quad (7.21)$$

Thus substituting (7.16) into (7.14) and using (7.20) and (7.21), we find that

$$\tilde{\nabla}^2 h^2 + h^i\tilde{\nabla}_i h^2 = 2\tilde{\nabla}^i h^j\tilde{\nabla}_i h_j + 2\tilde{\nabla}^i(dh)_{ij}h^j + 4\partial_i q^M\partial_j q^N g_{MN}h^ih^j . \quad (7.22)$$

This expression is close to the one required for the maximum principle to apply. It remains to determine dh . For this, consider the jk -component of the 3-form field equation to find

$$\nabla^i(v_{\underline{r}}H_{ijk} + x_{\underline{r}}^I H_{ijk}^I) = \epsilon_{ijkl}\partial^i v_{\underline{r}}h^l + v_{\underline{r}}\epsilon_{ijkl}\nabla^i h^l = 0 , \quad (7.23)$$

which implies that

$$dh = 0 , \quad (7.24)$$

Substituting this into (7.22), we get (7.10).

7.2.2 Supersymmetry enhancement

To demonstrate supersymmetry enhancement for the backgrounds with $h \neq 0$, let us re-investigate the KSEs for the fields given in (7.4). It is straightforward to see by substituting (7.4) into the KSEs that the general form of a Killing spinor is

$$\epsilon = \epsilon_+ + \epsilon_- = \eta_+ - \frac{u}{2}h_i\Gamma^i\Gamma_+\eta_- + \eta_- , \quad \Gamma_{\pm}\eta_{\pm} = \Gamma_{\pm}\epsilon_{\pm} = 0 , \quad (7.25)$$

where η_{\pm} depend only on the coordinates of \mathcal{S} . In addition the gravitino KSE requires that

$$\hat{\nabla}_i \epsilon + \mathcal{C}_i^{r'} \rho_{r'} \epsilon = 0 , \quad (7.26)$$

the tensorini KSEs implies that

$$(1 \pm \frac{1}{2}) T_i^I \Gamma^i \epsilon_{\pm} - \frac{1}{12} H_{ijk}^I \Gamma^{ijk} \epsilon_{\pm} = 0 , \quad (7.27)$$

and the hyperini KSEs gives

$$i \Gamma^i \epsilon_{\pm j} V_i^{aj} = 0 . \quad (7.28)$$

Next we shall show that both

$$\epsilon_1 = 1 + e_{1234} , \quad \epsilon_2 = \Gamma_- h_i \Gamma^i (1 + e_{1234}) - u k^2 (1 + e_{1234}) , \quad (7.29)$$

are Killing spinors, where we have set $k^2 = h^2$ for the constant length of h . Observe that the second Killing spinor is constructed by setting $\eta_+ = 0$ and $\eta_- = \Gamma_- h_i \Gamma^i (1 + e_{1234})$.

We have already solved the KSEs for ϵ_1 . Next observe that ϵ_2 solves the gravitino KSE as the Clifford algebra operation $h_i \Gamma^i \Gamma_-$ commutes with the supercovariant derivative in (7.26) as a consequence of the reduction of holonomy demonstrated in the previous section. In addition, the same Clifford operation commutes with the hyperini KSE as a result of the second eqn in (7.11) and (7.28).

It remains to show that ϵ_2 solves the tensorini KSE as well. This is a consequence of (7.21). For this observe that the metric induced on $SO(1, n_T)/SO(n_T)$ by the algebraic equation $\eta_{rs} v^r v^s = 1$ is the standard hyperbolic metric. So it has definite signature and as a result,

$$\partial_i v^i = 0 . \quad (7.30)$$

Thus, we conclude that the scalar fields are constant and the 3-form field strengths of the tensorini multiplet vanish. This agrees with the classification results of [20] for solutions of the KSEs of 6-dimensional supergravity preserving at least two supersymmetries whose Killing spinors have compact isotropy group and reviewed in section 5.3. Some of the results of this section are tabulated in table 3.

7.3 Geometry

To investigate the geometry of spacetime, one can compute the form bi-linears associated with the Killing spinors (7.29). In particular, one finds that the spacetime admits 3 $\hat{\nabla}$ -parallel 1-forms given by

$$\lambda^- = \mathbf{e}^- , \quad \lambda^+ = \mathbf{e}^+ - \frac{1}{2} k^2 u^2 \mathbf{e}^- - u h , \quad \lambda^1 = k^{-1} (h + k^2 u \mathbf{e}^-) . \quad (7.31)$$

Moreover, the Lie algebra of the associated vector fields closes in $\mathfrak{sl}(2, \mathbb{R})$. To verify this, see [67]. Since h is $\hat{\nabla}$ -parallel, the spacetime is locally metrically a product $SL(2, \mathbb{R}) \times \Sigma^3$, ie

$$ds^2 = ds^2(SL(2, \mathbb{R})) + ds^2(\Sigma^3) ,$$

$\text{Iso}(\eta_+)$	$\text{hol}(\tilde{\mathcal{D}})$	N	η_+
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	$Sp(1)$	2	$1 + e_{1234}$
$Sp(1) \cdot U(1) \ltimes \mathbb{H}$	$U(1)$	4	$1 + e_{1234}, i(1 - e_{1234})$
$Sp(1) \ltimes \mathbb{H}^4$	$\{1\}$	8	$1 + e_{1234}, i(1 - e_{1234}), e_{12} - e_{34}, i(e_{12} + e_{34})$

Table 3. Some of the geometric data used to solving the gravitino KSE are described. In the first column, we give the isotropy groups, $\text{Iso}(\eta_+)$, of $\{\eta_+\}$ spinors in $Spin(5, 1) \cdot Sp(1)$. In the second column we state the holonomy of the supercovariant connection $\tilde{\mathcal{D}}$ of the horizon section \mathcal{S} in each case. The holonomy of $\hat{\tilde{\nabla}}$ is identical to that of $\hat{\nabla}$. In the third column, we present the number of \mathcal{D} -parallel spinors and in the last column we give representatives of the $\{\eta_+\}$ spinors.

$$\begin{aligned} H &= d\text{vol}(SL(2, \mathbb{R})) + d\text{vol}(\Sigma^3) , \\ q^M &= q^M(z) , \end{aligned} \quad (7.32)$$

where the scalars of the hyper-multiplet depend only on the coordinates z of Σ^3 .

In addition to the 1-forms given in (7.31), the spacetime admits 3 more twisted 1-forms bilinears, see [20] and section 5.3. For the Killing spinors (7.29), these are given by

$$e^{r'} = k^{-1} h_j (J^{r'})^j_i \mathbf{e}^i , \quad (7.33)$$

where $J^{r'}$ is a quaternionic structure on \mathcal{S} associated with the twisted Hermitian 2-forms (5.8).

Observe that the frame $e^{r'}$ is orthogonal to h and the rotation between the \mathbf{e}^i and $(h, e^{r'})$ is in $SO(4)$. Therefore $(k^{-1}h, e^{r'})$ is another frame on \mathcal{S} with $e^{r'}$ adapted to Σ^3 . Thus $ds^2(\mathcal{S}) = k^{-2}h^2 + ds^2(\Sigma^3)$ with $ds^2(\Sigma^3) = \delta_{r's'} e^{r'} e^{s'}$.

The metric on Σ^3 is restricted by the Einstein equation (7.13) and the integrability condition (7.8). The former gives

$$R_{r's'}^{(3)} - \frac{1}{2} k^2 \delta_{r's'} - 2 \partial_{r'} q^M \partial_{s'} q^N g_{MN} = 0 , \quad (7.34)$$

where r', s' are indices of Σ^3 and $R^{(3)}$ is the Ricci tensor of Σ^3 . This is an equation which determines the metric on Σ^3 in terms of h and the hyper-multiplet scalars q . The integrability condition (7.8) does not give an independent condition on the metric of Σ^3 .

It remains to find the restriction imposed by supersymmetry on the scalars q of the hyper-multiplet. Using the results of section 5.3, equation (5.55) gives

$$\partial_{r'} q^M = -\epsilon_{r'}{}^{s't'} (I_{s'})^M_N \partial_{t'} q^N . \quad (7.35)$$

Constant maps are solutions from Σ^3 into the scalar manifold \mathcal{Q} of the hyper-multiplet scalars are solutions.

The geometry on Σ^3 is determined by (7.34) and depends on the solutions of (7.35). For the constant solutions of (7.35), Σ^3 is locally isometric to S^3 equipped with the round metric, and so the near horizon geometry is $AdS_3 \times S^3$.

Next suppose the existence of non-trivial solutions for the equation (7.35), and upon substitution the existence of solutions for (7.34). An priori one expects that the geometry on Σ^3 depends on the choice of Quaternionic Kähler manifold \mathcal{Q} for the hyper-multiplets

and the choice of a solution of (7.35). However, the differential structure on Σ^3 is independent of these choices. To show this first observe that the Ricci tensor $R^{(3)}$ is strictly positive. This turns out to be sufficient to determine the topology on Σ^3 . To see this note that in 3 dimensions the Ricci tensor determines the curvature of a manifold. Next, the strict positivity of the Ricci tensor implies that the (reduced) holonomy of the Levi-Civita connection of Σ^3 is $SO(3)$. Then a result of Gallot and Meyer, see [75], implies that Σ^3 is a homology 3-sphere. A brief proof of this is as follows. Since the holonomy of the Levi-Civita connection of Σ^3 is $SO(3)$, the only parallel forms are the constant real maps and the volume form of the manifold. On the other hand, the positivity of the Riemann curvature tensor implies that all harmonic forms are parallel and the fundamental group is finite. Thus de Rham cohomology of Σ^3 coincides with that of S^3 and so Σ^3 is a homology 3-sphere. In addition since the fundamental group is finite, the universal cover of Σ^3 is compact and so by the Poincaré conjecture [76] homeomorphic, and so diffeomorphic, to the 3-sphere.

8 N=4 and N=8 horizons

8.1 N=4 horizons

We have shown that if $h \neq 0$, the near horizon geometries preserve 2, 4 or 8 supersymmetries. We have already investigated the case with 2 supersymmetries. The two additional Killing spinors of horizons with 4 supersymmetries can be chosen as

$$\epsilon^3 = i(1 - e_{1234}) , \quad \epsilon^4 = -ik^2 u(1 - e_{1234}) + ih_i \Gamma^{+i}(1 - e_{1234}) . \quad (8.1)$$

These horizons are examples of $N = 4$ supersymmetric backgrounds with compact isotropy group investigated in section 6.1. Observe that $\epsilon^3 = \rho^1 \epsilon^1$ and $\epsilon^4 = \rho^1 \epsilon^2$. Thus the KSEs must commute with ρ^1 . As a result ω_1 is a well-defined Hermitian form on \mathcal{S} . The 1-form $\hat{\nabla}$ -parallel spinor bilinears are

$$\begin{aligned} \lambda^- &= \mathbf{e}^- , & \lambda^+ &= \mathbf{e}^+ - \frac{1}{2} k^2 u^2 \mathbf{e}^- - u h , & \lambda^1 &= k^{-1} (h + k^2 u \mathbf{e}^-) , \\ \lambda^4 &= e^1 , \end{aligned} \quad (8.2)$$

where the first 3 bilinears are those of horizons with two supersymmetries and e^1 is given in (7.33). The associated vector fields are Killing and their Lie algebra is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$.

The spacetime is locally metrically a product $AdS_3 \times \Sigma^3$, as for horizons preserving 2 supersymmetries. In addition in this case, Σ^3 is locally a S^1 fibration over a 2-dimensional manifold Σ^2 . The fibre direction is spanned by $\lambda^4 = e^1$. Thus

$$ds^2(\Sigma^3) = (e^1)^2 + ds^2(\Sigma^2) , \quad ds^2(\mathcal{S}) = k^{-2} h^2 + (e^1)^2 + ds^2(\Sigma^2) . \quad (8.3)$$

Observe that $de^1 \neq 0$ as $e^1 \wedge de^1$ is proportional to $\tilde{H} = d\text{vol}(\Sigma^3)$, and so the fibration is twisted.

It remains to specify the topology of Σ^2 . For this first observe that from the results of [20] and of section 6.1.4, the hyper-multiplet scalars depend only on the coordinates of Σ^2 . Then using (7.34), one finds that the Ricci tensor of Σ^2 is positive and so Σ^2 is a topological sphere. Finally the hyperini KSE implies that q are pseudo-holomorphic maps from Σ^2 into the Quaternionic Kähler manifold \mathcal{Q} .

8.2 N=8 horizons

As in the cases with 2 and 4 supersymmetries, one can show that the spacetime is locally $AdS_3 \times \Sigma^3$. In addition for horizons with 8 supersymmetries, the hyperini KSE implies that the scalars of the hyper-multiplet are constant, see [20] and section 6.2. This is compatible with the assertion made in the attractor mechanism, see [77] for the 6-dimensional supergravity case, that all the scalars take constant values at the horizon. In such case, the Einstein equation implies that Σ^3 is locally isometric to S^3 . Thus the only near horizon geometry preserving 8 supersymmetries with $h \neq 0$ is $AdS_3 \times S^3$.

9 (1,0)-superconformal theories

As another application, spinorial geometry will be used to investigate the brane solitons of the KSEs of 6-dimensional superconformal field theories. A consequence of AdS/CFT correspondence [10] is that the field theory dual of M-theory on the $AdS_7 \times S^4$ background is a (2,0) superconformal theory in six dimensions which describes a multiple M5-brane system. So far an action for such a theory has not been constructed which is local and 6D Lorentz covariant, though there have been suggestions [78, 79, 80] which either preserve a subset of the required symmetries or do not have a general gauge group because of the rigidity in the existence of Euclidean 3-Lie algebras [81, 82]. In fact it is not apparent that such the (2,0) theory has a classical action as it does not have a coupling constant and so a small coupling expansion. Nevertheless if such a theory exists it has to pass several consistency checks, see eg [83]. These include that after compactification on a circle one should recover the maximally supersymmetric gauge theory which describes D5-branes and it should also have a self-dual string and a 3-brane solitons which are dictated from the M-brane intersection rules. These state that a M2-brane ends on a M5-brane on a self-dual string and that two M5-branes intersect on a 3-brane [37, 38]. It is expected from the perspective of a M5-brane theory that the locus of these intersections manifest as worldvolume solitons. The effective dynamics of a single M5-brane has been described in [84, 85, 86].

Following a similar strategy to multiple M2-branes [87, 88] where worldvolume theories were considered preserving less than maximal supersymmetry [89], the authors of [35, 36] suggested a class of (1,0) superconformal theories with general gauge groups. Some of these models admit local actions [35, 36, 90] but suffer from several pathologies which include the non existence of a ground state and possibly the presence of negative norm states. Nevertheless in addition to the classical superconformal invariance and general gauge group, as we shall show, exhibit brane solitons in accordance to the M-brane intersection rules, and an intricate mathematical structure [91].

The application of the spinorial geometry to (1,0) superconformal theories leads to a systematic solution of their KSEs and to the construction of explicit self-dual string and 3-brane solitons [33, 34]. The string solutions are smooth because they are regularized by the size of instantons.

9.1 (1,0) superconformal theory and KSEs

9.1.1 Fields and KSEs

The (1,0) superconformal models constructed in [35, 36] have vector, tensor and hyper-multiplets as well as appropriate higher form fields which appear in Stuckelberg-type of couplings. The field content of the vector multiplets is $(A_\mu^r, \lambda^{ir}, Y^{ijr})$, where r labels the different vector multiplets and $i, j = 1, 2$ are the $Sp(1)$ R-symmetry indices, A_μ^r are 1-form gauge potentials, λ^{ir} are symplectic Majorana-Weyl spinors and Y^{ijr} are auxiliary fields. The field content of the tensor multiplets is $(\phi^I, \chi^{iI}, B_{\mu\nu}^I)$, where I labels the different tensor multiplets, ϕ^I are scalars, χ^{iI} are symplectic Majorana-Weyl spinors, of opposite chirality from those of the vector multiplets, and $B_{\mu\nu}^I$ are the 2-form gauge potentials. The field content of the hyper-multiplets are (q^M, ψ^a) , where q^M are the hyper-multiplet scalars, which are maps from the spacetime to a hyper-Kähler cone. The latter requires some explanation. Supersymmetry in rigidly supersymmetric theories requires that the hyper-multiplet scalars take values on a hyper-Kähler manifold \mathcal{Q} instead of a Quaternionic Kähler one that appears in supergravity. In addition, the existence of superconformal symmetry further restricts the hyper-Kähler manifold to admit a homothetic motion associated with a potential. This is because conformal invariance requires that all fields have a definite scaling dimension. As a result, this makes the hyper-Kähler manifold locally a hyper-Kähler cone. ψ^a are symplectic Majorana-Weyl spinors of the same chirality as χ^{iI} .

The field strengths of the 1- and 2-form gauge potentials associated with the vector and tensor multiplets are

$$\mathcal{F}_{\mu\nu}^r \equiv 2\partial_{[\mu}A_{\nu]}^r - f_{st}{}^r A_\mu^s A_\nu^t + h_I^r B_{\mu\nu}^I, \quad (9.1)$$

$$\mathcal{H}_{\mu\nu\rho}^I \equiv 3D_{[\mu}B_{\nu\rho]}^I + 6d_{rs}^I A_{[\mu}^r \partial_\nu A_{\rho]}^s - 2f_{pq}{}^s d_{rs}^I A_{[\mu}^r A_\nu^p A_{\rho]}^q + g^{Ir} C_{\mu\nu\rho r}, \quad (9.2)$$

respectively, where $f_{rs}{}^t$, h_I^r , g^{Ir} and $d_{rs}^I = d_{(rs)}^I$ are coupling constants, and $C_{\mu\nu\rho r}$ are three-form gauge potentials introduced via a Stückelberg-type of coupling. In addition,

$$D_\mu \Lambda^s \equiv \partial_\mu \Lambda^s + A_\mu^r (X_r)_t{}^s \Lambda^t, \quad D_\mu \Lambda^I \equiv \partial_\mu \Lambda^I + A_\mu^r (X_r)_J{}^I \Lambda^J, \quad (9.3)$$

where X_r are given by

$$(X_r)_t{}^s = -f_{rt}{}^s + d_{rt}^I h_I^s, \quad (X_r)_J{}^I = 2h_J^s d_{rs}^I - g^{Is} b_{Jsr}. \quad (9.4)$$

The various coupling satisfy a long list

$$\begin{aligned} 2(d_{r(u}^J d_{v)s}^I - d_{rs}^I d_{uv}^J) h^s{}_J &= 2f_{r(u}{}^s d_{v)s}^I - b_{Jsr} d_{uv}^J g^{Is}, \\ (d_{rs}^J b_{Iut} + d_{rt}^J b_{Isu} + 2d_{ru}^K b_{Kst} \delta_I^J) h_J^u &= f_{rs}{}^u b_{Iut} + f_{rt}{}^u b_{Isu} + g^{Ju} b_{Iur} b_{Jst}, \\ f_{[pq}{}^u f_{r]u}{}^s - \frac{1}{3} h_I^s d_{u[p}^I f_{qr]}^u &= 0, \\ h_I^r g^{Is} &= 0, \\ f_{rs}{}^t h_I^r - d_{rs}^J h_J^t h_I^r &= 0, \\ g^{Js} h_K^r b_{Isr} - 2h_I^s h_K^r d_{rs}^J &= 0, \\ -f_{rt}{}^s g^{It} + d_{rt}^J h_J^s g^{It} - g^{It} g^{Js} b_{Jtr} &= 0. \end{aligned} \quad (9.5)$$

of restrictions required by gauge invariance established in [35]. In addition, these models are described by an action provided there is a maximally split signature metric³ η_{IJ} such that

$$g^{Ir} = \eta^{IJ} h_I^r, \quad d_{rt}^I = \frac{1}{2} \eta^{IJ} b_{Jrt}. \quad (9.6)$$

From now on, the indices I, J are raised and lowered with η .

To couple hyper-multiplets to the above system [36], one assumes that the hyper-Kähler cone \mathcal{Q} admits tri-holomorphic isometries generated by the vector fields $X_{(\mathfrak{m})} = X_{(\mathfrak{m})}^M \partial_M$, ie isometries which leave also the three complex structures of the hyper-Kähler space invariant. Typically only some of the vector multiplets will be gauged. For this, introduce the embedding tensor $\theta_r^{\mathfrak{m}}$ and define

$$A^{\mathfrak{m}} = A^r \theta_r^{\mathfrak{m}}, \quad \lambda^{\mathfrak{m}} = \lambda^r \theta_r^{\mathfrak{m}}, \quad Y_{ij}^{\mathfrak{m}} = Y_{ij}^r \theta_r^{\mathfrak{m}}, \quad (9.7)$$

where for consistency with the gauge transformations

$$h_I^r \theta_r^{\mathfrak{m}} = 0, \quad f_{rs}^t \theta_t^{\mathfrak{m}} = \theta_r^{\mathfrak{n}} \theta_s^{\mathfrak{p}} f_{\mathfrak{np}}^{\mathfrak{m}}, \quad (9.8)$$

and where $[X_{(\mathfrak{n})}, X_{(\mathfrak{p})}] = -f_{\mathfrak{np}}^{\mathfrak{m}} X_{(\mathfrak{m})}$. The KSEs of the model, which are the vanishing conditions for the supersymmetry transformations of the fermions evaluated at the locus where all fermions vanish, are

$$\begin{aligned} \delta \lambda^{ir} &= \frac{1}{8} \mathcal{F}_{\mu\nu}^r \gamma^{\mu\nu} \epsilon^i - \frac{1}{2} Y^{ijr} \epsilon_j + \frac{1}{4} h_I^r \phi^I \epsilon^i = 0, \\ \delta \chi^{iI} &= \frac{1}{48} \mathcal{H}_{\mu\nu\rho}^I \gamma^{\mu\nu\rho} \epsilon^i + \frac{1}{4} D_\mu \phi^I \gamma^\mu \epsilon^i = 0, \\ \delta \psi^{\mathfrak{a}} &= \frac{1}{2} D_\mu q^M \gamma^\mu \epsilon_{\mathfrak{i}} E^{\mathfrak{ia}}{}_M = 0, \end{aligned} \quad (9.9)$$

where

$$D_\mu q^M = \partial_\mu q^M - A_\mu^{\mathfrak{m}} X_{(\mathfrak{m})}^M. \quad (9.10)$$

In addition, $E_{\mathfrak{ia}}^M$ is the symplectic frame of the hyper-Kähler cone, ie the hyper-Kähler metric and hypercomplex structure are given as

$$g_{MN} = \epsilon_{ij} \epsilon_{\mathfrak{ab}} E^{\mathfrak{ia}}{}_M E^{\mathfrak{jb}}{}_N, \quad (I_\tau)^M{}_N = -i (\sigma_\tau)^{\mathfrak{i}}{}_{\mathfrak{j}} \delta^{\mathfrak{a}}{}_{\mathfrak{b}} E_{\mathfrak{ia}}^M E_N^{\mathfrak{jb}}, \quad (9.11)$$

where ϵ_{ij} and $\epsilon_{\mathfrak{ab}}$ are the symplectic (fundamental) forms of $Sp(1)$ and $Sp(n)$, respectively, and $\sigma_\tau, \tau = 1, 2, 3$ are the Pauli matrices. In analogy with similar variations in 6-dimensional (1,0) supergravity, we refer to these KSEs as the gaugini, tensorini and hyperini KSEs, respectively.

The Lagrangian for these theories consist of two parts. One part, \mathcal{L}_{VT} , involves the vector and tensor multiplets, and the second part, \mathcal{L}_H , contains the hyper-multiplets. These two parts are independently supersymmetric and the supersymmetry transformation of the vector multiplets used in the coupling of the hyper-multiplets in \mathcal{L}_H is obtained by contraction with the embedding tensor.

³Since the metric is maximally split, the kinetic energy of some of the fields is negative which may lead to ghosts in the spectrum. This is an issue affecting this class of theories.

9.1.2 Field equations

The field equations of the system are

$$\begin{aligned}
D^\mu D_\mu \phi^I &= -\frac{1}{2} d_{rs}^I (\mathcal{F}_{\mu\nu}^r \mathcal{F}^{\mu\nu s} - 4Y_{ij}^r Y^{ijs}) - 3d_{rs}^I h_J^r h_K^s \phi^J \phi^K , \\
b_{Irs} Y_{ij}^s \phi^I &= \frac{1}{2\lambda} \theta_r^{\mathfrak{m}} \mu_{\mathfrak{m}ij} , \\
b_{Irs} \mathcal{F}_{\mu\nu}^s \phi^I &= \frac{1}{4!} \epsilon_{\mu\nu\lambda\rho\sigma\tau} \mathcal{H}_r^{(4)\lambda\rho\sigma\tau} , \\
g_{MN} \nabla_\mu D^\mu q^N &= -Y_{ij}^{\mathfrak{m}} \partial_M \mu_{(\mathfrak{m})}^{ij} ,
\end{aligned} \tag{9.12}$$

where

$$\nabla_\mu D^\mu q^M = \partial_\mu D^\mu q^M + \Gamma_{NP}^M D^\mu q^N D_\mu q^P - \partial_N X_{(\mathfrak{m})}^M \theta_r^{\mathfrak{m}} A_\mu^r D^\mu q^N , \tag{9.13}$$

λ is a constant, and $\mu_{(\mathfrak{m})\tau}$,

$$X_{(\mathfrak{m})}^N (\omega_\tau)_{NM} = -\partial_M \mu_{(\mathfrak{m})\tau} , \quad (\omega_\tau)_{MN} = g_{MP} (I_\tau)^P{}_N , \tag{9.14}$$

are the moment maps. Observe that generically the theory has a cubic scalar field interaction and so the potential term is not bounded from below. These field equations are also supplemented with the Bianchi identities

$$\begin{aligned}
D_{[\mu} \mathcal{F}_{\nu\rho]}^r &= \frac{1}{3} h_I^r \mathcal{H}_{\mu\nu\rho}^I , \\
D_{[\mu} \mathcal{H}_{\nu\rho\sigma]}^I &= \frac{3}{2} d_{rs}^I \mathcal{F}_{[\mu\nu}^r \mathcal{F}_{\rho\sigma]}^s + \frac{1}{4} g^{Ir} \mathcal{H}_{\mu\nu\rho\sigma}^{(4)} , \\
D_{[\mu} \mathcal{H}_{\nu\lambda\rho\sigma]}^{(4)} &= -4d_{Irs} \mathcal{F}_{[\mu\nu}^s \mathcal{H}_{\lambda\rho\sigma]}^I + \frac{1}{5} \theta_r^{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}\mu\nu\lambda\rho\sigma}^{(5)} ,
\end{aligned} \tag{9.15}$$

where $\mathcal{H}_{\mu\nu\rho\sigma}^{(4)}$ is the field strength of the 3-form, and the duality relations

$$\frac{1}{5!} \epsilon_{\mu\nu\rho\lambda\sigma\tau} \theta_r^{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(5)\nu\rho\lambda\sigma\tau} = (X_r)_{IJ} \phi^I D_\mu \phi^J + \frac{2}{\lambda} \theta_r^{\mathfrak{m}} X_{(\mathfrak{m})M} D_\mu q^M , \tag{9.16}$$

ie the 5-form field strength is dual to the hyper-multiplet scalars.

9.1.3 KSEs revisited

The KSEs of the system are the vanishing conditions of the supersymmetry variations of the fermions given in (9.9). These KSEs are very similar to the (1,0) supergravity KSEs. The only differences are that there is no gravitino KSE and there is some relabeling of the fields, ie there are three instead of four KSEs the gaugini, tensorini and hyperini ones. Because of this, they can be rewritten in a basis where the symplectic Majorana-Weyl spinors are identified with the $SU(2)$ Majorana-Weyl spinors of $Spin(9,1)$ as in (3.16). In particular, the KSEs can now be rewritten as

$$\frac{1}{4} \mathcal{F}_{\mu\nu}^r \gamma^{\mu\nu} \epsilon + (Y^r)_{r'} \rho^{r'} \epsilon + \frac{1}{2} h_I^r \phi^I \epsilon = 0 , \tag{9.17}$$

$$\frac{1}{12} \mathcal{H}_{\mu\nu\rho}^I \gamma^{\mu\nu\rho} \epsilon + D_\mu \phi^I \gamma^\mu \epsilon = 0 , \tag{9.18}$$

$$\frac{1}{2} D_\mu q^M \gamma^\mu \epsilon_i E^{\text{ia}}{}_M = 0 , \tag{9.19}$$

where we have set

$$-Y^{ijr}\epsilon_j = (Y^r)_{r'}\rho^{r'}\epsilon^i, \quad (9.20)$$

and it is understood that

$$\epsilon_1 = -\epsilon^2, \quad \epsilon_2 = \Gamma_{34}\epsilon^1, \quad (9.21)$$

and where ρ 's are given in (3.17). The latter identification applies in the context of hyperini KSE.

9.1.4 Solution of KSEs

To solve the KSEs, it is essential to note that the spinorial geometry method is not sensitive to the way that the components of the KSEs in the Clifford algebra expansion depend on the physical fields. Since the KSEs of the (1,0) superconformal theory (9.17), (9.18) and (9.19) have the same lexicographic structure as the gaugini, tensorini and hyperini KSEs of (1,0) supergravity (3.18), the method developed to solve the latter also applies to solve the former. In fact, the analysis is simpler than that of the supergravity theory as one does not have to solve the gravitino KSE. The results are summarized in two tables. In table 4, the isotropy groups of the Killing spinors in $Spin(5,1) \cdot Sp(1)$ are given and a choice of representatives for the invariant spinors, while in table 5 the number of supersymmetries preserved in each case is denoted. Note that for the hyperini KSE there is a distinct case preserving 3 Killing spinors. The Killing spinors can be chosen as in (4.7).

N	Isotropy Groups	Invariant Spinors
1	$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}$
2	$(Sp(1) \cdot U(1)) \ltimes \mathbb{H}$	$1 + e_{1234}, i(1 - e_{1234})$
4	$Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}, i(1 - e_{1234}), e_{12} - e_{34}, i(e_{12} + e_{34})$
2	$Sp(1)$	$1 + e_{1234}, e_{15} + e_{2345}$
4	$U(1)$	$1 + e_{1234}, i(1 - e_{1234}), e_{15} + e_{2345}, i(e_{15} - e_{2345})$

Table 4: The first column gives the number of invariant spinors, the second column the associated isotropy groups and the third column representatives of the invariant spinors. Observe that if 3 spinors are invariant, then there is a fourth one which is also invariant under the same isotropy group. Moreover the isotropy group of more than 4 linearly independent spinors is the identity.

Having identified the Killing spinors and the fractions of supersymmetry preserved, it is straightforward to derive the linear system in each case and solve it to find the conditions on the fields required by supersymmetry. Since the spacetime is flat, the task is rather straightforward and it follows closely the analysis we have already presented for supergravity. So instead of repeating the details, only the final result will be stated in each case with a minimum explanation.

Isotropy Groups	Gaugini	Tensorini	Hyperini
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	1	4	1
$Sp(1) \cdot U(1) \ltimes \mathbb{H}$	2	4	2
$Sp(1) \ltimes \mathbb{H}$	4	4	3, 4
$Sp(1)$	2	8	2
$U(1)$	4	8	4
$\{1\}$	8	8	8

Table 5: In the first column the isotropy groups of the Killing spinors of the gaugini KSE are given. In the second, third and fourth columns the number of Killing spinors of the gaugini, tensorini and hyperini KSEs are stated, respectively. The isotropy groups of the Killing spinors of the tensorini KSE are either $Sp(1) \ltimes \mathbb{H}$ or $\{1\}$. The cases that do not appear in the table do not independently occur.

9.1.5 N=1 solutions

As in the case of supergravity, the KSEs can be easily expressed after choosing a light-cone Hermitian coordinate system for the 6-dimensional Minkowski spacetime metric. In particular, one writes

$$ds^2 = 2e^-e^+ + \delta_{ij}e^ie^j = e^-e^+ + 2\delta_{\alpha\bar{\beta}}e^\alpha e^{\bar{\beta}} = 2dx^+dx^- + 2\delta_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}} , \quad (9.22)$$

and assumes the apparent identification between the frame $(e^+, e^-, e^\alpha, e^{\bar{\alpha}})$ which appears in supergravity and the coordinates $(x^+, x^-, z^\alpha, z^{\bar{\alpha}})$ of the Minkowski spacetime.

The solution of the gaugini KSEs (9.17) can be expressed as

$$\mathcal{F}^r = -h_I^r \phi^I e^- \wedge e^+ + \mathcal{F}_{-i}^r e^- \wedge e^i + (Y^r)_{s'} \omega^{s'} + \mathcal{F}^{\text{asd}, r} , \quad (9.23)$$

where $\omega_{s'}$ are the twisted Hermitian forms in (5.8).

Similarly, the tensorini KSEs (9.18) give

$$\begin{aligned} \mathcal{H}^I &= \frac{1}{2} \mathcal{H}_{-ij}^I e^- \wedge e^i \wedge e^j - D_i \phi^I e^- \wedge e^+ \wedge e^i + \frac{1}{3!} D_\ell \phi^I \epsilon_{ijk}^\ell e^i \wedge e^j \wedge e^k , \\ D_+ \phi^I &= 0 , \end{aligned} \quad (9.24)$$

where \mathcal{H}_{-ij}^I is *anti-self-dual* in the directions transverse to (e^+, e^-) . Unlike the gaugini KSEs, the tensorini KSEs exhibit supersymmetry enhancement. In particular, if they admit one Killing spinor ϵ , they also admit three additional Killing spinors given by $\rho^1 \epsilon, \rho^2 \epsilon$ and $\rho^3 \epsilon$. For $\epsilon = 1 + e_{1234}$, all four Killing spinors are given by the $Sp(1) \ltimes \mathbb{H}$ invariant spinors of table 4.

Next the hyperini KSE gives that

$$D_+ q^M = 0 , \quad (\mathcal{I}^i)^M{}_N D_i q^N = 0 , \quad (9.25)$$

where $(\mathcal{I}^i) = (I_\tau, 1_{4n \times 4n})$ and I_τ have been given in (9.11).

9.1.6 $N = 2$ solutions with non-compact isotropy group

The solution to the gaugini KSEs can be expressed as

$$\mathcal{F}^r = -h_I^r \phi^I e^- \wedge e^+ + \mathcal{F}_{-i}^r e^- \wedge e^i + Y^r \omega + \mathcal{F}^{\text{asd},r} , \quad (9.26)$$

where we have set $Y^r = (Y^r)_1$. In this case, $(Y^r)_2 = (Y^r)_3 = 0$. As we have explained in the previous section, the tensorini KSEs give the same conditions as in the $N = 1$ case.

The conditions imposed by the hyperini KSEs on the fields can be expressed as

$$D_+ q^M = 0 , \quad D_i q^N (I_3)^M{}_N = J^j{}_i D_j q^M , \quad (9.27)$$

where I_3 is defined in (9.11) and $J^i{}_j = (i\delta^\alpha_\beta, -i\delta^{\bar{\alpha}}_{\bar{\beta}})$. In the absence of gauge fields, the above condition becomes the Cauchy-Riemann equation and q is a holomorphic map from the transverse space to the (e^+, e^-) to the hyper-Kähler cone \mathcal{Q} with respect to the indicated pair of complex structures. The choice complex structures depends on the choice of representatives for the Killing spinors.

9.1.7 $N = 2$ solutions with compact isotropy group

From the analysis of the supergravity KSEs, we know that the spacetime admits a 3+3 split. This split can be expressed by splitting the spacetime index as $\mu = (a, i)$, where $a = +, -, 1$ and i labels the remaining three coordinates, ie the metric is written as

$$ds^2 = \eta_{ab} e^a e^b + \delta_{ij} e^i e^j = \eta_{ab} dx^a dx^b + \delta_{ij} dx^i dx^j . \quad (9.28)$$

In this notation, the gaugino KSEs give

$$\mathcal{F}^r = -\varepsilon_{ijk} (Y^r)^k e^i \wedge e^j , \quad h_I^r \phi^I = 0 , \quad (9.29)$$

where we have appropriately identified the spacetime index with that which labels the auxiliary fields Y .

The tensorini KSEs imply that

$$\mathcal{H}_{\mu\nu\rho}^I = 0 , \quad D_\mu \phi^I = 0 . \quad (9.30)$$

Clearly in this case, the tensorini KSEs preserve all 8 supersymmetries. Moreover, the integrability of the last condition in (9.30) implies that

$$F_{\mu\nu}^r X_r J^I \phi^J = 0 , \quad (9.31)$$

where $F_{\mu\nu}^r = 2\partial_{[\mu} A_{\nu]}^r + X_{st}{}^r A_\mu^s A_\nu^t$.

Finally, the hyperini KSEs give

$$D_a q^M = 0 , \quad D_i q^M = -\epsilon_i{}^{jk} (I_j)^M{}_N D_k q^N , \quad (9.32)$$

as in section 5.3.

9.1.8 N=4 solutions with non-compact isotropy group

The gaugini KSEs give

$$\mathcal{F}^r = -h_I^r \phi^I e^- \wedge e^+ + \mathcal{F}_{-i}^r e^- \wedge e^i + \mathcal{F}^{\text{asd},r}, \quad (9.33)$$

where now $Y^1 = Y^2 = Y^3 = 0$. The tensorini KSE gives the same conditions as those in the $N = 1$ case. It remains to solve the hyperini KSE. This gives that the only non-vanishing component is $D_- q^M$.

9.1.9 N=3 Non-Compact

The hyperini KSE admits a special case that preserves 3 supersymmetries. The conditions for this are

$$D_+ q^M = 0, \quad (J_\tau)^j{}_i D_j q^M = (I_\tau)^M{}_N D_i q^N, \quad \tau = 1, 2, 3, \quad (9.34)$$

for an appropriate choice of a hypercomplex structure J_τ in the directions transverse to (e^+, e^-) . Therefore in the absence of gauge couplings, the hyper-scalars are quaternionic maps. Clearly, the directions transverse to (e^+, e^-) can be identified with the quaternions \mathbb{H} . If the Obata curvature of the the hyper-Kähler cone vanishes, then it is possible to introduce quaternionic coordinates on the hyper-Kähler cone. In such a case q 's can be written as quaternions \mathbf{q} and (9.34) implies that $\mathbf{q} = \mathbf{q}(\mathbf{x}, x^-)$, $\mathbf{x} \in \mathbb{H}$.

9.1.10 N=4 solutions with compact isotropy group

The spacetime admits a 4+2 split. The metric can be written as

$$ds^2 = \eta_{ab} e^a e^b + \delta_{ij} e^i e^j = \eta_{ab} dx^a dx^b + 2dz^2 dz^{\bar{2}}, \quad (9.35)$$

ie the spacetime index $\mu = (a, i) = (a, 2, \bar{2})$ The tensorini KSEs give that

$$\mathcal{H}_{\mu\nu\rho} = D_\mu \phi = 0. \quad (9.36)$$

The gaugini KSEs imply

$$\mathcal{F}^r = -2iY^r e^2 \wedge e^{\bar{2}}, \quad h_I^r \phi^I = 0, \quad (9.37)$$

where we have set $Y^r = (Y^r)_1$.

Next the hyperini KSEs give

$$D_a q^M = 0, \quad J^j{}_i D_j q^M = (I_3)^M{}_N D_i q^N, \quad (9.38)$$

where $J^i{}_j = (i\delta^2{}_2, -i\delta^{\bar{2}}{}_2)$.

9.1.11 Maximally supersymmetric solutions

As we have mentioned all backgrounds which preserve more than 4 supersymmetries are maximally supersymmetric. It is straightforward to see that the conditions on the fluxes for maximally supersymmetric backgrounds are

$$D_\mu \phi^I = 0, \quad h_I^r \phi^I = 0, \quad \mathcal{F}_{\mu\nu}^r = 0, \quad \mathcal{H}_{\mu\nu\rho}^I = 0, \quad Y^{\text{ij}r} = 0, \quad D_\mu q^M = 0. \quad (9.39)$$

Thus all the scalars ϕ^I and q^M are covariantly constant. In addition, those projected by h are required to vanish. Similarly the 2-form and 3-form field strengths vanish as well. The same applies for the auxiliary fields Y .

9.2 Self-dual string solitons

9.2.1 A class of models

A large class of models has been constructed in [35, 36] by considering a Lie algebra \mathfrak{g} and a representation \mathcal{R} . The bosonic fields of the vector and tensor multiplets are chosen as

$$A^r = (A^{\mathfrak{m}}, A^A), \quad Y^r = (Y^{\mathfrak{m}}, Y^A), \quad B^I = (B^A, B_A), \quad \phi^I = (\phi^A, \phi_A), \quad (9.40)$$

ie A and Y take values in $\mathfrak{g} \oplus \mathcal{R}$ while B and ϕ take values in $\mathcal{R} \oplus \mathcal{R}^*$. Moreover the non-vanishing couplings are chosen as

$$\begin{aligned} \eta^A_B &= \eta_B^A = \delta_B^A, \quad h^B_A = g_A^B = \delta_A^B, \quad f_{\mathfrak{m}A}^B = -\frac{1}{2}(T_{\mathfrak{m}})_A^B, \quad f_{\mathfrak{m}\mathfrak{n}}^{\mathfrak{p}}, \\ d_{\mathfrak{m}A}^B &= \frac{1}{2}b_{A\mathfrak{m}}^B = \frac{1}{2}b_{\mathfrak{m}A}^B = \frac{1}{2}(T_{\mathfrak{m}})_A^B, \quad d_{ABC} = d_{(ABC)} = b_{BCA}, \\ d_{AB\mathfrak{m}} &= d_{(AB)\mathfrak{m}} = \frac{1}{2}b_{AB\mathfrak{m}} = \frac{1}{2}b_{A\mathfrak{m}B}, \quad d_{A\mathfrak{m}\mathfrak{n}}, \quad b_{A(\mathfrak{m}\mathfrak{n})} = 2d_{A(\mathfrak{m}\mathfrak{n})}, \quad \theta_{\mathfrak{m}}^{\mathfrak{n}} = \delta_{\mathfrak{m}}^{\mathfrak{n}} \end{aligned} \quad (9.41)$$

where $T_{\mathfrak{m}}$ are the representation matrices of \mathfrak{g} in \mathcal{R} . These solve all the constraints on the couplings imposed on these models provided that $d_{\mathfrak{m}AB}, d_{\mathfrak{m}\mathfrak{n}A}$ and d_{ABC} are invariant under the action of \mathfrak{g} .

9.2.2 Self-dual string solitons from instantons

Motivated from the M-brane intersection rules, we shall seek self-dual string solitons in the class of models described in the previous section which preserve 1/2 of the supersymmetry. The relevant class of supersymmetric backgrounds for self-dual string solitons are those with 4 Killing spinors that have isotropy group $Sp(1) \ltimes \mathbb{H}$ in table 4. The conditions on the fields of the vector and tensor multiplets are given in [33] and in section 3.6 for the hyper-multiplet scalars. Similar solutions have been found in [33] for another class of models, see also [92]. The self-dual string soliton on a single M5-brane has been found in [93] and it is singular at the position of the string.

To solve the supersymmetry conditions, Bianchi identities and field equations, suppose that the fields have support on 4-directions transverse to the light-cone coordinates (x^+, x^-) which are identified with the world-sheet of the string. In addition choose

$$\mathcal{F}^r = (\mathcal{F}^{\mathfrak{m}}, 0), \quad \mathcal{H}^I = (0, \mathcal{H}_A), \quad \phi^I = (0, \phi_A), \quad \mathcal{H}_r^{(4)} = Y^r = \mathcal{H}^{(5)} = 0, \quad (9.42)$$

with \mathcal{F}^r purely magnetic. We focus on models for which the only non-vanishing coupling constants with all indices lowered are $b_{A\mathfrak{m}\mathfrak{n}}, d_{A\mathfrak{m}\mathfrak{n}}$. In addition we assume that either the model is not coupled to hyper-multiplets or if it is coupled, then the hyper-scalars are at a maximally supersymmetric vacuum for consistency, ie the gauging and the hyper-Kähler cone has been chosen such that there is a value $q = q_0$ and

$$\mu_{\mathfrak{m}}(q_0) = 0, \quad \partial_M \mu_{\mathfrak{m}}(q_0) = 0, \quad (9.43)$$

where μ are the moment maps defined in (9.14). For the flat hyperkähler cone, such a value is $q_0 = 0$ or any other fixed point of rotational isometries that are gauged. In either case, the contribution from the hyper-multiplets decouples.

The remaining non-trivial Bianchi identities and field equations that one has to demonstrate are

$$D_{[\mu}\mathcal{F}_{n\ell]}^{\mathfrak{m}} = 0 \ , \quad b^B{}_{An}\mathcal{F}_{m\ell}^n\phi_B = 0 \ , \quad d^B{}_{An}\mathcal{F}_{[\mu\nu]}^n\mathcal{H}_{\lambda\rho\sigma]B} = 0 \ , \quad (9.44)$$

and

$$D_{[\mu}\mathcal{H}_{\nu\rho\sigma]A} = \frac{3}{2}d_{Amn}\mathcal{F}_{[\mu\nu]}^{\mathfrak{m}}\mathcal{F}_{\rho\sigma]}^n \ , \ D_m D^m \phi_A = -\frac{1}{2}d_{Amn}\mathcal{F}_{mn}^{\mathfrak{m}}\mathcal{F}^{mn} \ . \quad (9.45)$$

These conditions can be solved provided that \mathcal{R} can be decomposed as $\mathcal{R} = I \oplus \mathcal{R}'$, where I is a trivial representation of \mathfrak{g} and take that ϕ_A and \mathcal{H}_A vanish unless they lie along the trivial representation, and denote the non-vanishing fields with ϕ_0 and \mathcal{H}_0 , respectively. Such a choice will solve the last two conditions in (9.44) as $T_{\mathfrak{m}}$ vanishes along the trivial representation. The first condition in (9.44) is solved by identifying $\mathcal{F}^{\mathfrak{m}}$ with the field strength of a gauge field with Lie algebra \mathfrak{g} .

It remains to solve the conditions in (9.45). First observe that $D_m D^m \phi_0 = \partial_m \partial^m \phi_0$, and similarly on \mathcal{H}_0 , and identify d_{0mn} with a bi-invariant metric on \mathfrak{g} . Next set

$$\mathcal{H}_0 = -\partial_i \phi_0 dx^- \wedge dx^+ \wedge dx^i + \frac{1}{3!} \partial_j \phi_0 \epsilon^j{}_{i_1 i_2 i_3} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \ , \quad (9.46)$$

Then recall that the KSEs for $Sp(1) \ltimes \mathbb{H}$ invariant spinors require that \mathcal{F} is an anti-self dual instanton. Because of this and (9.46), the second condition in (9.45) implies the first. Finally, the last condition in (9.45) is solved because the Pontryagin form of instantons can be written as the Laplacian on a scalar function [94]. In addition for generic values of instanton moduli space, all the string solutions are smooth.

To present an explicit solution take $\mathfrak{g} = \mathfrak{su}(2)$, we consider the configuration with instanton number 1 and use the results of [95]. In such a case, the gauge connection A of $\mathcal{F}^{\mathfrak{m}} = \mathcal{F}^{ab}$ and ϕ_0 can be written as

$$\begin{aligned} A^{ab} &= 2(J^{r'})^{ab}(J_{r'})_{ij} \frac{x^j}{|x|^2 + \rho^2} e^i \ , \quad \phi_0 = c + 4\sqrt{2} \frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2} + h_0 \ , \\ h_0 &= \sum_{\nu} \frac{Q_{\nu}}{|x - x_{\nu}|^2} \ , \end{aligned} \quad (9.47)$$

where x are the coordinates in $\mathbb{R}^{5,1}$ transverse to string worldsheet coordinates (x^+, x^-) , c is a constant, and ρ is the instanton modulus. Moreover h_0 is a multi-centred harmonic function, which if it is included in the solution, then delta function sources have to be added in the field equation for ϕ_0 . Let us focus on the solution with $h_0 = 0$. Such a solution is smooth at a generic value of ρ . At large $|x|$, ie far away from the string, the scalar ϕ_0 converges to the constant c , and the gauge connection is a pure gauge. As $|x|$ becomes small, the values of ϕ_0 and A are regulated by the modulus $\rho \neq 0$ of the instanton. In particular at $|x| = 0$, the value of ϕ_0 is $c + (8\sqrt{2}/\rho^2)$. Assuming that the theory describes a M5-brane, c becomes the position of M5 at infinity. Then the M5-brane is “pulled” by the M2-branes ending on it and its position shifts by $8\sqrt{2}/\rho^2$. Of course as the instanton size becomes small, $\rho^2 \rightarrow 0$, a throat is developed. This solution becomes similar to self-dual strings of [93].

The dyonic string charge q_s of all solutions can be computed by integrating \mathcal{H}_0 on the 3-sphere at infinity. After an appropriate normalization, this can be identified with the instanton number k , ie

$$q_s = \int_{S^3 \subset \mathbb{R}^4} \mathcal{H}_0 = k . \quad (9.48)$$

All solutions with any instanton number k are smooth at a generic point in the instanton moduli space.

9.3 3-branes

Motivated from the M-brane intersection rules which state that two M5-branes intersect on a 3-brane, we shall describe a class of models which exhibit 3-brane solitons. These are those for which all the potentials vanish and the only active fields are those of the hyper-multiplets. Moreover, the hyper-multiplet scalars depend only on the two transverse directions to the 3-brane soliton. First to identify the models with 3-brane solitons suppose that the hyper-multiplets are not gauged, ie the embedding tensor $\theta = 0$. Moreover set all the fields apart from the hypermultiplet scalars q and $\mathcal{H}^{(5)}$ equal to zero. The only non-trivial conditions that have to be satisfied to construct solutions are the field equations for q and the hyperini KSEs.

To solve the hyperini KSEs, we take the case with 4 supersymmetries and compact isotropy group. The relevant equations are given in (9.38). The solution of KSEs implies that the hyper-multiplet scalars do not depend on four directions, as expected for a 3-brane soliton, and (9.38) is a Cauchy-Riemann equations which implies that q is a holomorphic curve into the hyper-Kähler cone. In addition, the field equation for the q 's is automatically satisfied.

Utilizing the $N = 2$ solutions with compact isotropy group, a similar argument reveals the existence of string solitons preserving 1/4 of supersymmetry supported by a holomorphic surface embedded into the hyper-Kähler cone. It is expected that such solitons are associated with a triple M5-brane intersection on a string.

10 Conclusions

A distinct role amongst the solutions of a supersymmetric theory have those that preserve some of the supercharges. Such solutions apart from the field equations also solve the KSEs of supersymmetric theories. In the context of string theory, M-theory and supergravity such solutions have found widespread applications to compactifications, black holes, AdS/CFT, and branes. They have also been instrumental in understanding string dualities. The systematic investigation of supersymmetric solutions is an outstanding problem and is instrumental in the development of various aspects of string and M-theory as these require a deeper understanding of such solutions. Apart from the applications to physics, there are notable applications to geometry as intricate geometric structures arise in the description of such solutions.

Spinorial geometry provides a general framework to understand the solution of the KSEs of supersymmetric systems. It has been used to systematically solve the KSEs

of heterotic supergravity, the KSEs of $D = 4$ $\mathcal{N} = 1$ supergravity and those of (1,0) 6-dimensional supergravity to determine both the fractions of supersymmetry preserved and the geometries of all backgrounds. It can also be used to solve the KSEs of supersymmetric theories for a small or near maximal number of supersymmetries.

In this review, the spinorial geometry method has been described as it applies in the 6-dimensional (1,0) supergravity. It has been explained how all the fractions of supersymmetry preserved by the supersymmetric backgrounds have been identified as well as how the KSEs can be solved to determine the conditions on the fields and the spacetime geometry. In addition two applications have been presented. One is on the near horizon geometries of 6-dimensional black holes. In particular, it is explained how a class of such horizons is locally a product $AdS_3 \times \Sigma^3$. Another application is on the description of the brane solitons of 6-dimensional (1,0) superconformal theories. In particular a systematic description of all configurations that preserve a fraction of supersymmetry is given.

The applicability of spinorial geometry is not limited to six dimensions. It can be applied to supersymmetric systems in all dimensions providing a systematic way to identify the supersymmetric backgrounds. It is expected that in the next few years a clear picture will emerge of the geometry of all such solutions. Applications will include insights into the backgrounds used in AdS/CFT, the discovery of new black holes in various dimensions and the unraveling of their symmetries, the understanding of brane solutions and their intersections, and the exploration of superconformal theories.

Acknowledgements

I would like to thank the Albert-Einstein-Institute, Max Planck Institute in Golm, for providing a stimulating environment to complete this review. I am partially supported by the STFC grant ST/J002798/1.

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